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Mathematical Analysis II

Springer

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Title of Russian edition:
Matematicheskij Analiz (Part II, 4th corrected edition, Moscow, 2002)
MCCME (Moscow Center for Continuous Mathematical Education Publ.)

Cataloging-in-Publication Data applied for
A catalog record for this book is available from the Library of Congress.
Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at http://dnb.ddb.de

Mathematics Subject Classification (2000): Primary 00A05
Secondary: 26-01, 40-01, 42-01, 54-01, 58-01

ISBN 3-540-40633-6 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag is a part of Springer Science+Business Media

springeronline.com

Cover design: *design & production* GmbH, Heidelberg
Typeset by the translator using a Springer LaTeX macro package
Printed on acid-free paper 46/3111- 5 4 3 2 1 SPIN 11605140

Prefaces

Preface to the English Edition

An entire generation of mathematicians has grown up during the time between the appearance of the first edition of this textbook and the publication of the fourth edition, a translation of which is before you. The book is familiar to many people, who either attended the lectures on which it is based or studied out of it, and who now teach others in universities all over the world. I am glad that it has become accessible to English-speaking readers.

This textbook consists of two parts. It is aimed primarily at university students and teachers specializing in mathematics and natural sciences, and at all those who wish to see both the rigorous mathematical theory and examples of its effective use in the solution of real problems of natural science.

The textbook exposes classical analysis as it is today, as an integral part of Mathematics in its interrelations with other modern mathematical courses such as algebra, differential geometry, differential equations, complex and functional analysis.

The two chapters with which this second book begins, summarize and explain in a general form essentially all most important results of the first volume concerning continuous and differentiable functions, as well as differential calculus. The presence of these two chapters makes the second book formally independent of the first one. This assumes, however, that the reader is sufficiently well prepared to get by without introductory considerations of the first part, which preceded the resulting formalism discussed here. This second book, containing both the differential calculus in its generalized form and integral calculus of functions of several variables, developed up to the general formula of Newton–Leibniz–Stokes, thus acquires a certain unity and becomes more self-contained.

More complete information on the textbook and some recommendations for its use in teaching can be found in the translations of the prefaces to the first and second Russian editions.

Moscow, 2003 *V. Zorich*

Preface to the Fourth Russian Edition

In the fourth edition all misprints that the author is aware of have been corrected.

Moscow, 2002 *V. Zorich*

Preface to the Third Russian Edition

The third edition differs from the second only in local corrections (although in one case it also involves the correction of a proof) and in the addition of some problems that seem to me to be useful.

Moscow, 2001 *V. Zorich*

Preface to the Second Russian Edition

In addition to the correction of all the misprints in the first edition of which the author is aware, the differences between the second edition and the first edition of this book are mainly the following. Certain sections on individual topics – for example, Fourier series and the Fourier transform – have been recast (for the better, I hope). We have included several new examples of applications and new substantive problems relating to various parts of the theory and sometimes significantly extending it. Test questions are given, as well as questions and problems from the midterm examinations. The list of further readings has been expanded.

Further information on the material and some characteristics of this second part of the course are given below in the preface to the first edition.

Moscow, 1998 *V. Zorich*

Preface to the First Russian Edition

The preface to the first part contained a rather detailed characterization of the course as a whole, and hence I confine myself here to some remarks on the content of the second part only.

The basic material of the present volume consists on the one hand of multiple integrals and line and surface integrals, leading to the generalized Stokes' formula and some examples of its application, and on the other hand the machinery of series and integrals depending on a parameter, including

Fourier series, the Fourier transform, and the presentation of asymptotic expansions.

Thus, this Part 2 basically conforms to the curriculum of the second year of study in the mathematics departments of universities.

So as not to impose rigid restrictions on the order of presentation of these two major topics during the two semesters, I have discussed them practically independently of each other.

Chapters 9 and 10, with which this book begins, reproduce in compressed and generalized form, essentially all of the most important results that were obtained in the first part concerning continuous and differentiable functions. These chapters are starred and written as an appendix to Part 1. This appendix contains, however, many concepts that play a role in any exposition of analysis to mathematicians. The presence of these two chapters makes the second book formally independent of the first, provided the reader is sufficiently well prepared to get by without the numerous examples and introductory considerations that, in the first part, preceded the formalism discussed here.

The main new material in the book, which is devoted to the integral calculus of several variables, begins in Chapter 11. One who has completed the first part may begin the second part of the course at this point without any loss of continuity in the ideas.

The language of differential forms is explained and used in the discussion of the theory of line and surface integrals. All the basic geometric concepts and analytic constructions that later form a scale of abstract definitions leading to the generalized Stokes' formula are first introduced by using elementary material.

Chapter 15 is devoted to a similar summary exposition of the integration of differential forms on manifolds. I regard this chapter as a very desirable and systematizing supplement to what was expounded and explained using specific objects in the mandatory Chapters 11–14.

The section on series and integrals depending on a parameter gives, along with the traditional material, some elementary information on asymptotic series and asymptotics of integrals (Chap. 19), since, due to its effectiveness, the latter is an unquestionably useful piece of analytic machinery.

For convenience in orientation, ancillary material or sections that may be omitted on a first reading, are starred.

The numbering of the chapters and figures in this book continues the numbering of the first part.

Biographical information is given here only for those scholars not mentioned in the first part.

As before, for the convenience of the reader, and to shorten the text, the end of a proof is denoted by $\square$. Where convenient, definitions are introduced by the special symbols $:=$ or $=:$ (equality by definition), in which the colon stands on the side of the object being defined.

Continuing the tradition of Part 1, a great deal of attention has been paid to both the lucidity and logical clarity of the mathematical constructions themselves and the demonstration of substantive applications in natural science for the theory developed.

Moscow, 1982 *V. Zorich*

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9 *Continuous Mappings (General Theory)

In this chapter we shall generalize the properties of continuous mappings established earlier for numerical-valued functions and mappings of the type $f : \mathbb{R}^m \to \mathbb{R}^n$ and discuss them from a unified point of view. In the process we shall introduce a number of simple, yet important concepts that are used everywhere in mathematics.

9.1 Metric Spaces

9.1.1 Definition and Examples

Definition 1. A set X is said to be endowed with a *metric* or *a metric space structure* or to be a *metric space* if a function

$$d : X \times X \to \mathbb{R} \tag{9.1}$$

is exhibited satisfying the following conditions:

a) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$,

b) $d(x_1, x_2) = d(x_2, x_2)$ (symmetry),

c) $d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3)$ (the triangle inequality),

where x_1, x_2, x_3 are arbitrary elements of X.

In that case, the function (9.1) is called a *metric* or *distance* on X.

Thus a *metric space* is a pair (X, d) consisting of a set X and a metric defined on it.

In accordance with geometric terminology the elements of X are called *points*.

We remark that if we set $x_3 = x_1$ in the triangle inequality and take account of conditions a) and b) in the definition of a metric, we find that

$$0 \le d(x_1, x_2) ,$$

that is, a distance satisfying axioms a), b), and c) is nonnegative.

Let us now consider some examples.

Example 1. The set $\mathbb{R}$ of real numbers becomes a metric space if we set $d(x_1, x_2) = |x_2 - x_1|$ for any two numbers x_1 and x_2, as we have always done.

Example 2. Other metrics can also be introduced on $\mathbb{R}$. A trivial metric, for example, is the discrete metric in which the distance between any two distinct points is 1.

The following metric on $\mathbb{R}$ is much more substantive. Let $x \mapsto f(x)$ be a nonnegative function defined for $x \geq 0$ and vanishing for $x = 0$. If this function is strictly convex upward, then, setting

$$d(x_1, x_2) = f(|x_1 - x_2|) \tag{9.2}$$

for points $x_1, x_2 \in \mathbb{R}$, we obtain a metric on $\mathbb{R}$.

Axioms a) and b) obviously hold here, and the triangle inequality follows from the easily verified fact that f is strictly monotonic and satisfies the following inequalities for $0 < a < b$:

$$f(a + b) - f(b) < f(a) - f(0) = f(a).$$

In particular, one could set $d(x_1, x_2) = \sqrt{|x_1 - x_2|}$ or $d(x_1, x_2) = \frac{|x_1 - x_2|}{1+|x_1 - x_2|}$. In the latter case the distance between any two points of the line is less than 1.

Example 3. Besides the traditional distance

$$d(x_1, x_2) = \sqrt{\sum_{i=1}^{n} |x_1^i - x_2^i|^2} \tag{9.3}$$

between points $x_1 = (x_1^1, \dots, x_1^n)$ and $x_2 = (x_2^1, \dots, x_2^n)$ in $\mathbb{R}^n$, one can also introduce the distance

$$d_p(x_1, x_2) = \left(\sum_{i=1}^{n} |x_1^i - x_2^i|^p \right)^{1/p}, \tag{9.4}$$

where $p \geq 1$. The validity of the triangle inequality for the function (9.4) follows from Minkowski's inequality (see Subsect. 5.4.2).

Example 4. When we encounter a word with incorrect letters while reading a text, we can reconstruct the word without too much trouble by correcting the errors, provided the number of errors is not too large. However, correcting the error and obtaining the word is an operation that is sometimes ambiguous. For that reason, other conditions being equal, one must give preference to the interpretation of the incorrect text that requires the fewest corrections.

Accordingly, in coding theory the metric (9.4) with $p = 1$ is used on the set of all finite sequences of length n consisting of zeros and ones.

Geometrically the set of such sequences can be interpreted as the set of vertices of the unit cube $I = \{x \in \mathbb{R}^n | 0 \le x^i \le 1, i = 1, \dots, n\}$ in $\mathbb{R}^n$. The distance between two vertices is the number of interchanges of zeros and ones needed to obtain the coordinates of one vertex from the other. Each such interchange represents a passage along one edge of the cube. Thus this distance is the shortest path along the edges of the cube from one of the vertices under consideration to the other.

Example 5. In comparing the results of two series of n measurements of the same quantity the metric most commonly used is (9.4) with $p = 2$. The distance between points in this metric is usually called their *mean-square deviation.*

Example 6. As one can easily see, if we pass to the limit in (9.4) as $p \to +\infty$, we obtain the following metric in $\mathbb{R}^n$:

$$d(x_1, x_2) = \max_{1 \le i \le n} |x_1^i - x_2^i| . \tag{9.5}$$

Example 7. The set $C[a, b]$ of functions that are continuous on a closed interval becomes a metric space if we define the distance between two functions f and g to be

$$d(f, g) = \max_{a \le x \le b} |f(x) - g(x)| . \tag{9.6}$$

Axioms a) and b) for a metric obviously hold, and the triangle inequality follows from the relations

$$|f(x) - h(x) \le |f(x) - g(x)| + |g(x) - h(x)| \le d(f, g) + d(g, h) ,$$

that is,

$$d(f, h) = \max_{a \le x \le b} |f(x) - h(x)| \le d(f, g) + d(g, h) .$$

The metric (9.6) – the so-called *uniform* or *Chebyshev* metric in $C[a, b]$ – is used when we wish to replace one function by another (for example, a polynomial) from which it is possible to compute the values of the first with a required degree of precision at any point $x \in [a, b]$. The quantity $d(f, g)$ is precisely a characterization of the precision of such an approximate computation.

The metric (9.6) closely resembles the metric (9.5) in $\mathbb{R}^n$.

Example 8. Like the metric (9.4), for $p \ge 1$ we can introduce in $C[a, b]$ the metric

$$d_p(f, g) = \left(\int_a^b |f - g|^p(x)\, dx \right)^{1/p} . \tag{9.7}$$

It follows from Minkowski's inequality for integrals, which can be obtained from Minkowski's inequality for the Riemann sums by passing to the limit, that this is indeed a metric for $p \geq 1$.

The following special cases of the metric (9.7) are especially important: $p = 1$, which is the integral metric; $p = 2$, the metric of mean-square deviation; and $p = +\infty$, the uniform metric.

The space $C[a, b]$ endowed with the metric (9.7) is often denoted $C_p[a, b]$. One can verify that $C_\infty[a, b]$ is the space $C[a, b]$ endowed with the metric (9.6).

Example 9. The metric (9.7) could also have been used on the set $\mathcal{R}[a, b]$ of Riemann-integrable functions on the closed interval $[a, b]$. However, since the integral of the absolute value of the difference of two functions may vanish even when the two functions are not identically equal, axiom a) will not hold in this case. Nevertheless, we know that the integral of a nonnegative function $\varphi \in \mathcal{R}[a, b]$ equals zero if and only if $\varphi(x) = 0$ at almost all points of the closed interval $[a, b]$.

Therefore, if we partition $\mathcal{R}[a, b]$ into equivalence classes of functions, regarding two functions in $\mathcal{R}[a, b]$ as equivalent if they differ on at most a set of measure zero, then the relation (9.7) really does define a metric on the set $\widetilde{\mathcal{R}}[a, b]$ of such equivalence classes. The set $\widetilde{\mathcal{R}}[a, b]$ endowed with this metric will be denoted $\widetilde{\mathcal{R}}_p[a, b]$ and sometimes simply by $\mathcal{R}_p[a, b]$.

Example 10. In the set $C^{(k)}[a, b]$ of functions defined on $[a, b]$ and having continuous derivatives up to order k inclusive one can define the following metric:

$$d(f, g) = \max\{M_0, \dots, M_k\} , \tag{9.8}$$

where

$$M_i = \max_{a \leq x \leq b} |f^{(i)}(x) - g^{(i)}(x)| , \quad i = 0, 1, \dots, k .$$

Using the fact that (9.6) is a metric, one can easily verify that (9.8) is also a metric.

Assume for example that f is the coordinate of a moving point considered as a function of time. If a restriction is placed on the allowable region where the point can be during the time interval $[a, b]$ and the particle is not allowed to exceed a certain speed, and, in addition, we wish to have some assurance that the accelerations cannot exceed a certain level, it is natural to consider the set $\{\max\limits_{a \leq x \leq b} |f(x)|, \max\limits_{a \leq x \leq b} |f'(x)|, \max\limits_{a \leq x \leq b} |f''(x)|\}$ for a function $f \in C^{(2)}[a, b]$ and using these characteristics, to regard two motions f and g as close together if the quantity (9.8) for them is small.

These examples show that a given set can be metrized in various ways. The choice of the metric to be introduced is usually controlled by the statement of the problem. At present we shall be interested in the most general properties of metric spaces, the properties that are inherent in all of them.

9.1.2 Open and Closed Subsets of a Metric Space

Let (X, d) be a metric space. In the general case, as was done for the case $X = \mathbb{R}^n$ in Sect. 7.1, one can also introduce the concept of a ball with center at a given point, open set, closed set, neighborhood of a point, limit point of a set, and so forth.

Let us now recall these concepts, which are basic for what is to follow.

Definition 2. For $\delta > 0$ and $a \in X$ the set

$$B(a, \delta) = \{x \in X \,|\, d(a, x) < \delta\}$$

is called the *ball with center* $a \in X$ *of radius* δ or the δ*-neighborhood of the point* a.

This name is a convenient one in a general metric space, but it must not be identified with the traditional geometric image we are familiar with in $\mathbb{R}^3$.

Example 11. The unit ball in $C[a, b]$ with center at the function that is identically 0 on $[a, b]$ consists of the functions that are continuous on the closed interval $[a, b]$ and whose absolute values are less than 1 on that interval.

Example 12. Let X be the unit square in $\mathbb{R}^2$ for which the distance between two points is defined to be the distance between those same points in $\mathbb{R}^2$. Then X is a metric space, while the square X considered as a metric space in its own right can be regarded as the ball of any radius $\rho \geq \sqrt{2}/2$ about its center.

It is clear that in this way one could construct balls of very peculiar shape. Hence the term ball should not be understood too literally.

Definition 3. A set $G \subset X$ is *open in the metric space* (X, d) if for each point $x \in G$ there exists a ball $B(x, \delta)$ such that $B(x, \delta) \subset G$.

It obviously follows from this definition that X itself is an open set in (X, d). The empty set $\varnothing$ is also open. By the same reasoning as in the case of $\mathbb{R}^n$ one can prove that a ball $B(a, r)$ and its exterior $\{x \in X : d(a, x) \geq r\}$ are open sets. (See Examples 3 and 4 of Sect. 7.1).

Definition 4. A set $F \subset X$ is *closed in* (X, d) if its complement $X \setminus F$ is open in (X, d).

In particular, we conclude from this definition that the *closed ball*

$$\widetilde{B}(a, r) := \{x \in X \,|\, d(a, x) \leq r\}$$

is a closed set in a metric space (X, d).

The following proposition holds for open and closed sets in a metric space (X, d).

Proposition 1. *a) The union* $\bigcup_{\alpha \in A} G_\alpha$ *of the sets in any system* $\{G_\alpha,\, \alpha \in A\}$ *of sets* G_α *that are open in* X *is an open set in* X.

b) The intersection $\bigcap_{i=1}^{n} G_i$ *of any finite number of sets that are open in* X *is an open set in* X.

a′) The intersection $\bigcap_{\alpha \in A} F_\alpha$ *of the sets in any system* $\{F_\alpha,\, \alpha \in A\}$ *of sets* F_α *that are closed in* X *is a closed set in* X.

b′) The union $\bigcup_{i=1}^{n} F_i$ *of any finite number of sets that are closed in* X *is a closed set in* X.

The proof of Proposition 1 is a verbatim repetition of the proof of the corresponding proposition for open and closed sets in $\mathbb{R}^n$, and we omit it. (See Proposition 1 in Sect. 7.1.)

Definition 5. An open set in X containing the point $x \in X$ is called a *neighborhood* of the point x in X.

Definition 6. Relative to a set $E \subset X$, a point $x \in X$ is called

an interior point of E if some neighborhood of it is contained in X,

an exterior point of E if some neighborhood of it is contained in the complement of E in X,

a boundary point of E if it is neither interior nor exterior to E (that is, every neighborhood of the point contains both a point belonging to E and a point not belonging to E).

Example 13. All points of a ball $B(a,r)$ are interior to it, and the set $C_X\widetilde{B}(a,r) = X \setminus \widetilde{B}(a,r)$ consists of the points exterior to the ball $B(a,r)$.

In the case of $\mathbb{R}^n$ with the standard metric d the *sphere* $S(a,r) := \{x \in \mathbb{R}^n \,|\, d(a,x) = r > 0\}$ is the set of boundary points of the ball $B(a,r)$.[1]

Definition 7. A point $a \in X$ is a *limit point* of the set $E \subset X$ if the set $E \cap O(a)$ is infinite for every neighborhood $O(a)$ of the point.

Definition 8. The union of the set E and the set of all its limit points is called the *closure* of the set E in X.

As before, the closure of a set $E \subset X$ will be denoted $\overline{E}$.

Proposition 2. *A set* $F \subset X$ *is closed in* X *if and only if it contains all its limit points.*

Thus

$$(F \text{ is closed in } X) \Longleftrightarrow (F = \overline{F} \text{ in } X);.$$

We omit the proof, since it repeats the proof of the analogous proposition for the case $X = \mathbb{R}^n$ discussed in Sect. 7.1.

[1] In connection with Example 13 see also Problem 2 at the end of this section.

9.1.3 Subspaces of a Metric space

If (X, d) is a metric space and E is a subset of X, then, setting the distance between two points x_1 and x_2 of E equal to $d(x_1, x_2)$, that is, the distance between them in X, we obtain the metric space (E, d), which is customarily called a *subspace* of the original space (X, d).

Thus we adopt the following definition.

Definition 9. A metric space (X_1, d_1) is a *subspace of the metric space* (X, d) if $X_1 \subset X$ and the equality $d_1(a, b) = d(a, b)$ holds for any pair of points a, b in X_1.

Since the ball $B_1(a, r) = \{x \in X_1 | \, d_1(a, x) < r\}$ in a subspace (X_1, d_1) of the metric space (X, d) is obviously the intersection

$$B_1(a, r) = X_1 \cap B(a, r)$$

of the set $X_1 \subset X$ with the ball $B(a, r)$ in X, it follows that every open set in X_1 has the form

$$G_1 = X_1 \cap G ,$$

where G is an open set in X, and every closed set F_1 in X_1 has the form

$$F_1 = X_1 \cap F ,$$

where F is a closed set in X.

It follows from what has just been said that the properties of a set in a metric space of being open or closed are relative properties and depend on the ambient space.

Example 14. The open interval $|x| < 1$, $y = 0$ of the x-axis in the plane $\mathbb{R}^2$ with the standard metric in $\mathbb{R}^2$ is a metric space (X_1, d_1), which, like any metric space, is closed as a subset of itself, since it contains all its limit points in X_1. At the same time, it is obviously not closed in $\mathbb{R}^2 = X$.

This same example shows that openness is also a relative concept.

Example 15. The set $C[a, b]$ of continuous functions on the closed interval $[a, b]$ with the metric (9.7) is a subspace of the metric space $\mathcal{R}_p[a, b]$. However, if we consider the metric (9.6) on $C[a, b]$ rather than (9.7), this is no longer true.

9.1.4 The Direct Product of Metric Spaces

If (X_1, d_1) and (X_2, d_2) are two metric spaces, one can introduce a metric d on the direct product $X_1 \times X_2$. The commonest methods of introducing a metric in $X_1 \times X_2$ are the following. If $(x_1, x_2) \in X_1 \times X_2$ and $(x_1', x_2') \in X_1 \times X_2$, one may set

$$d((x_1,x_2),(x_1',x_2')) = \sqrt{d_1^2(x_1,x_1') + d_2^2(x_2,x_2')}\ ,$$

or

$$d(x_1,x_2),(x_1',x_2')) = d_1(x_1,x_1') + d_2(x_2,x_2')\ ,$$

or

$$d((x_1,x_2),(x_1',x_2')) = \max\left\{d_1(x_1,x_2'), d_2(x_2,x_2')\right\}\ .$$

It is easy to see that we obtain a metric on $X_1 \times X_2$ in all of these cases.

Definition 10. if (X_1,d_1) and (X_2,d_2) are two metric spaces, the space $(X_1 \times X_2, d)$, where d is a metric on $X_1 \times X_2$ introduced by any of the methods just indicated, will be called the *direct product* of the original metric spaces.

Example 16. The space $\mathbb{R}^2$ can be regarded as the direct product of two copies of the metric space $\mathbb{R}$ with its standard metric, and $\mathbb{R}^3$ is the direct product $\mathbb{R}^2 \times \mathbb{R}^1$ of the spaces $\mathbb{R}^2$ and $\mathbb{R}^1 = \mathbb{R}$.

9.1.5 Problems and Exercises

1. a) Extending Example 2, show that if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function that is strictly convex upward and satisfies $f(0) = 0$, while (X,d) is a metric space, then one can introduce a new metric d_f on X by setting $d_f(x_1,x_2) = f\Big(d(x_1,x_2)\Big)$.

b) Show that on any metric space (X,d) one can introduce a metric $d'(x_1,x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}$ in which the distance between the points will be less than 1.

2. Let (X,d) be a metric space with the trivial (*discrete*) metric shown in Example 2, and let $a \in X$. For this case, what are the sets $B(a,1/2)$, $B(a,1)$, $\overline{B}(a,1)$, $\widetilde{B}(a,1)$, $B(a,3/2)$, and what are the sets $\{x \in X \,|\, d(a,x) = 1/2\}$, $\{x \in X \,|\, d(a,x) = 1\}$, $\overline{B}(a,1) \setminus B(a,1)$, $\widetilde{B}(a,1) \setminus B(a,1)$?

3. a) Is it true that the union of any family of closed sets is a closed set?

b) Is every boundary point of a set a limit point of that set?

c) Is it true that in any neighborhood of a boundary point of a set there are points in both the interior and exterior of that set?

d) Show that the set of boundary points of any set is a closed set.

4. a) Prove that if (Y,d_Y) is a subspace of the metric space (X,d_X), then for any open (resp. closed) set G_Y (resp. F_Y) in Y there is an open (resp. closed) set G_X (resp. F_X) in X such that $G_Y = Y \cap G_X$, (resp. $F_Y = Y \cap F_X$).

b) Verify that if the open sets G_Y' and G_Y'' in Y do not intersect, then the corresponding sets G_X' and G_X'' in X can be chosen so that they also have no points in common.

5. Having a metric d on a set X, one may attempt to define the distance $\tilde{d}(A,B)$ between sets $A \subset X$ and $B \subset X$ as follows:

$$\tilde{d}(A,B) = \inf_{a \in A,\, b \in B} d(a,b) .$$

a) Give an example of a metric space and two nonintersecting subsets of it A and B for which $\tilde{d}(A,B) = 0$.

b) Show, following Hausdorff, that on the set of closed sets of a metric space (X,d) one can introduce the *Hausdorff metric* D by assuming that for $A \subset X$ and $B \subset X$

$$D(A,B) := \max\left\{ \sup_{a \in A} \tilde{d}(a,B), \sup_{b \in B} \tilde{d}(A,b) \right\} .$$

9.2 Topological Spaces

For questions connected with the concept of the limit of a function or a mapping, what is essential in many cases is not the presence of any particular metric on the space, but rather the possibility of saying what a neighborhood of a point is. To convince oneself of that it suffices to recall that the very definition of a limit or the definition of continuity can be stated in terms of neighborhoods. Topological spaces are the mathematical objects on which the operation of passage to the limit and the concept of continuity can be studied in maximum generality.

9.2.1 Basic Definitions

Definition 1. A set X is said to be endowed with the structure of a *topological space* or a *topology* or is said to be a *topological space* if a system τ of subsets of X is exhibited (called *open sets in* X) possessing the following properties:

a) $\varnothing \in \tau$; $X \in \tau$.

b) $\big(\forall \alpha \in A\,(\tau_\alpha \in \tau)\big) \Longrightarrow \bigcup_{\alpha \in A} \tau_\alpha \in \tau$.

c) $\big(\tau_i \in \tau;\, i = 1, \dots, n\big) \Longrightarrow \bigcap_{i=1}^{n} \tau_i \in \tau$.

Thus, a topological space is a pair (X,τ) consisting of a set X and a system τ of distinguished subsets of the set having the properties that τ contains the empty set and the whole set X, the union of any number of sets of τ is a set of τ, and the intersection of any finite number of sets of τ is a set of τ.

As one can see, in the axiom system a), b), c) for a topological space we have postulated precisely the properties of open sets that we already proved in the case of a metric space. Thus any metric space with the definition of open sets given above is a topological space.

Thus *defining a topology* on X means exhibiting a system τ of subsets of X satisfying the axioms a), b), and c) for a topological space.

Defining a metric in X, as we have seen, automatically defines the topology on X induced by that metric. It should be remarked, however, that different metrics on X may generate the same topology on that set.

Example 1. Let $X = \mathbb{R}^n$ $(n > 1)$. Consider the metric $d_1(x_1, x_2)$ defined by relation (9.5) in Sect. 9.1, and the metric $d_2(x_1, x_2)$ defined by formula (9.3) in Sect. 9.1.

The inequalities

$$d_1(x_1, x_2) \le d_2(x_1, x_2) \le \sqrt{n} d_1(x_1, x_2) ,$$

obviously imply that every ball $B(a, r)$ with center at an arbitrary point $a \in X$, interpreted in the sense of one of these two metrics, contains a ball with the same center, interpreted in the sense of the other metric. Hence by definition of an open subset of a metric space, it follows that the two metrics induce the same topology on X.

Nearly all the topological spaces that we shall make active use of in this course are metric spaces. One should not think, however, that every topological space can be metrized, that is, endowed with a metric whose open sets will be the same as the open sets in the system τ that defines the topology on X. The conditions under which this can be done form the content of the so-called *metrization theorems.*

Definition 2. If (X, τ) is a topological space, the sets of the system τ are called the *open sets*, and their complements in X are called the *closed sets* of the topological space (X, τ).

A topology τ on a set X is seldom defined by enumerating all the sets in the system τ. More often the system τ is defined by exhibiting only a certain set of subsets of X from which one can obtain any set in the system τ through union and intersection. The following definition is therefore very important.

Definition 3. A *base of the topological space* (X, τ) (an *open base* or *base for the topology*) is a family $\mathfrak{B}$ of open subsets of X such that every open set $G \in \tau$ is the union of some collection of elements of the family $\mathfrak{B}$.

Example 2. If (X, d) is a metric space and (x, τ) the topological space corresponding to it, the set $\mathfrak{B} = \{B(a, r)\}$ of all balls, where $a \in X$ and $r > 0$, is obviously a base of the topology τ. Moreover, if we take the system $\mathfrak{B}$ of all balls with positive rational radii r, this system is also a base for the topology.

Thus a topology can be defined by describing only a base of that topology. As one can see from Example 2, a topological space may have many different bases for the topology.

Definition 4. The minimal cardinality among all bases of a topological space is called its *weight.*

As a rule, we shall be dealing with topological spaces whose topologies admit a countable base (see, however, Problems 4 and 6).

Example 3. If we take the system $\mathfrak{B}$ of balls in $\mathbb{R}^k$ of all possible rational radii $r = \frac{m}{n} > 0$ with centers at all possible rational points $\left(\frac{m_1}{n_1}, \ldots, \frac{m_k}{n_k}\right) \in \mathbb{R}^k$, we obviously obtain a countable base for the standard topology of $\mathbb{R}^k$. It is not difficult to verify that it is impossible to define the standard topology in $\mathbb{R}^k$ by exhibiting a finite system of open sets. Thus the standard topological space $\mathbb{R}^k$ has countable weight.

Definition 5. A *neighborhood* of a point of a topological space (X, τ) is an open set containing the point.

It is clear that if a topology τ is defined on X, then for each point the system of its neighborhoods is defined.

It is also clear that the system of all neighborhoods of all possible points of topological space can serve as a base for the topology of that space. Thus a topology can be introduced on X by describing the neighborhoods of the points of X. This is the way of defining the topology in X that was originally used in the definition of a topological space.[2] Notice, for example, that we have introduced the topology in a metric space itself essentially by saying what a δ-neighborhood of a point is. Let us give one more example.

Example 4. Consider the set $C(\mathbb{R}, \mathbb{R})$ of real-valued continuous functions defined on the entire real line. Using this set as foundation, we shall construct a new set – the set of germs of continuous functions. We shall regard two functions $f, g \in C(\mathbb{R}, \mathbb{R})$ as equivalent at the point $a \in \mathbb{R}$ if there is a neighborhood $U(a)$ of that point such that $\forall x \in U(a)\,\big(f(x) = g(x)\big)$. The relation just introduced really is an equivalence relation (it is reflexive, symmetric, and transitive). An equivalence class of continuous functions at the point $a \in \mathbb{R}$ is called *germ of continuous functions* at that point. If f is one of the functions generating the germ at the point a, we shall denote the germ itself by the symbol f_a. Now let us define a neighborhood of a germ. Let $U(a)$ be a neighborhood of the point a and f a function defined on $U(a)$ generating the germ f_a at a. This same function generates its germ f_x at any point $x \in U(a)$. The set $\{f_x\}$ of all germs corresponding to the points $x \in U(a)$ will be called a *neighborhood of the germ* f_a. Taking the set of such neighborhoods of all germs as the base of a topology, we turn the set of germs of continuous functions into a topological space. It is worthwhile to note that

[2] The concepts of a metric space and a topological space were explicitly stated early in the twentieth century. In 1906 the French mathematician M. Fréchet (1878–1973) introduced the concept of a metric space, and in 1914 the German mathematician F. Hausdorff (1868–1942) defined a topological space.

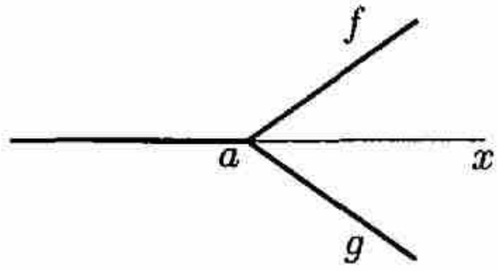

Fig. 9.1.

in the resulting topological space two different points (germs) f_a and g_a may not have disjoint neighborhoods (see Fig. 9.1)

Definition 6. A topological space is *Hausdorff* if the *Hausdorff axiom* holds in it: *any two distinct points of the space have nonintersecting neighborhoods.*

Example 5. Any metric space is obviously Hausdorff, since for any two points $a, b \in X$ such that $d(a,b) > 0$ their spherical neighborhoods $B\big(a, \frac{1}{2}d(a,b)\big)$, $B\big(b, \frac{1}{2}d(a,b)\big)$ have no points in common.

At the same time, as Example 4 shows, there do exist non-Hausdorff topological spaces. Perhaps the simplest example here is the topological space (X, τ) with the trivial topology $\tau = \{\varnothing, X\}$. If X contains at least two distinct points, then (X, τ) is obviously not Hausdorff. Moreover, the complement $X \setminus x$ of a point in this space is not an open set.

We shall be working exclusively with Hausdorff spaces.

Definition 7. A set $E \subset X$ is *(everywhere) dense* in a topological space (X, τ) if for any point $x \in X$ and any neighborhood $U(x)$ of it the intersection $E \cap U(X)$ is nonempty.

Example 6. If we consider the standard topology in $\mathbb{R}$, the set $\mathbb{Q}$ of rational numbers is everywhere dense in $\mathbb{R}$. Similarly the set $\mathbb{Q}^n$ of rational points in $\mathbb{R}^n$ is dense in $\mathbb{R}^n$.

One can show that in every topological space there is an everywhere dense set whose cardinality does not exceed the weight of the topological space.

Definition 8. A metric space having a countable dense set is called a *separable* space.

Example 7. The metric space $(\mathbb{R}^n, d)$ in any of the standard metrics is a separable space, since $\mathbb{Q}^n$ is dense in it.

Example 8. The metric space $\big(C([0,1], \mathbb{R}), d\big)$ with the metric defined by (9.6) is also separable. For, as follows from the uniform continuity of the functions $f \in C\big([0,1], \mathbb{R}\big)$, the graph of any such function can be approximated as closely as desired by a broken line consisting of a finite number of segments whose nodes have rational coordinates. The set of such broken lines is countable.

We shall be dealing mainly with separable spaces.

We now remark that, since the definition of a neighborhood of a point in a topological space is verbally the same as the definition of a neighborhood of a point in a metric space, the concepts of *interior point*, *exterior point*, *boundary point*, and *limit point* of a set, and the concept of the *closure* of a set, all of which use only the concept of a neighborhood, can be carried over without any changes to the case of an arbitrary topological space.

Moreover, as can be seen from the proof of Proposition 2 in Sect. 7.1, it is also true that a set in a Hausdorff space is closed if and only if it contains all its limit points.

9.2.2 Subspaces of a Topological Space

Let (X, τ_X) be a topological space and Y a subset of X. The topology τ_X makes it possible to define the following topology τ_Y in Y, called the *induced* or *relative* topology on $Y \subset X$.

We define an *open set in* Y to be any set G_Y of the form $G_Y = Y \cap G_X$, where G_X is an open set in X.

It is not difficult to verify that the system τ_Y of subsets of Y that arises in this way satisfies the axioms for open sets in a topological space.

As one can see, the definition of open sets G_Y in Y agrees with the one we obtained in Subsect. 9.1.3 for the case when Y is a subspace of a metric space X.

Definition 9. A subset $Y \subset X$ of a topological space (X, τ) with the topology τ_Y induced on Y is called a *subspace of the topological space* X.

It is clear that a set that is open in (Y, τ_Y) is not necessarily open in (X, τ_X).

9.2.3 The Direct Product of Topological Spaces

If (X_1, τ_1) and (X_2, τ_2) are two topological spaces with systems of open sets $\tau_1 = \{G_1\}$ and $\tau_2 = \{G_2\}$, we can introduce a topology on $X_1 \times X_2$ by taking as the base the sets of the form $G_1 \times G_2$.

Definition 10. The topological space $(X_1 \times X_2, \tau_1 \times \tau_2)$ whose topology has the base consisting of sets of the form $G_1 \times G_2$, where G_i is an open set in the topological space (X_i, τ_i), $i = 1, 2$, is called the *direct product* of the topological spaces (X_1, τ_1) and (X_2, τ_2).

Example 9. If $\mathbb{R} = \mathbb{R}^1$ and $\mathbb{R}^2$ are considered with their standard topologies, then, as one can see, $\mathbb{R}^2$ is the direct product $\mathbb{R}^1 \times \mathbb{R}^1$. For every open set in $\mathbb{R}^2$ can be obtained, for exmaple, as the union of "square" neighborhoods of all its points. And squares (with sides parallel to the axes) are the products of open intervals, which are open sets in $\mathbb{R}$.

It should be noted that the sets $G_1 \times G_2$, where $G_1 \in \tau_1$ and $G_2 \in \tau_2$, constitute only a base for the topology, not all the open sets in the direct product of topological spaces.

9.2.4 Problems and Exercises

1. Verify that if (X, d) is a metric space, then $\left(X, \frac{d}{1+d}\right)$ is also a metric space, and the metrics d and $\frac{d}{1+d}$ induce the same topology on X. (See also Problem 1 of the preceding section.)

2. a) In the set $\mathbb{N}$ of natural numbers we define a neighborhood of the number $n \in \mathbb{N}$ to be an arithmetic progression with difference d relatively prime to n. Is the resulting topological space Hausdorff?

b) What is the topology of $\mathbb{N}$, regarded as a subset of the set $\mathbb{R}$ of real numbers with the standard topology?

c) Describe all open subsets of $\mathbb{R}$.

3. If two topologies τ_1 and τ_2 are defined on the same set, we say that τ_2 is *stronger* than τ_1 if $\tau_1 \subset \tau_2$, that is τ_2 contains all the sets in τ_1 and some additional open sets not in τ_1.

a) Are the two topologies on $\mathbb{N}$ considered in the preceding problem comparable?

b) If we introduce a metric on the set $C[0,1]$ of continuous real-valued functions defined on the closed interval $[0,1]$ first by relation (9.6) of Sect. 9.1, and then by relation (9.7) of the same section, two topologies generally arise on $C[a,b]$. Are they comparable?

4. a) Prove in detail that the space of germs of continuous functions defined in Example 4 is not Hausdorff.

b) Explain why this topological space is not metrizable.

c) What is the weight of this space?

5. a) State the axioms for a topological space in the language of closed sets.

b) Verify that the closure of the closure of a set equals the closure of the set.

c) Verify that the boundary of any set is a closed set.

d) Show that if F is closed and G is open in (X, τ), then the set $G \setminus F$ is open in (X, τ).

e) If (Y, τ_Y) is a subspace of the topological space (X, τ), and the set E is such that $E \subset Y \subset X$ and $E \in \tau_X$, then $E \in \tau_Y$.

6. A topological space (X, τ) in which every point is a closed set is called a *topological space in the strong sense* or a τ_1-space. Verify the following statements.

a) Every Hausdorff space is a τ_1-space (partly for this reason, Hausdorff spaces are sometimes called τ_2-spaces).

b) Not every τ_1-space is a τ_2-space. (See Example 4).

c) The two-point space $X = \{a, b\}$ with the open sets $\{\varnothing, X\}$ is not a τ_1-space.

d) In a τ_1-space a set F is closed if and only if it contains all its limit points.

7. a) Prove that in any topological space there is an everywhere dense set whose cardinality does not exceed the weight of the space.

b) Verify that the following metric spaces are separable: $C[a,b]$, $C^{(k)}[a,b]$, $\mathcal{R}_1[a,b]$, $\mathcal{R}_p[a,b]$ (for the formulas giving the respective metrics see Sect. 9.1.)

c) Verify that if max is replaced by sup in relation (9.6) of Subsect. 9.1 and regarded as a metric on the set of all bounded real-valued functions defined on a closed interval $[a,b]$, we obtain a nonseparable metric space.

9.3 Compact Sets

9.3.1 Definition and General Properties of Compact Sets

Definition 1. A set K in a topological space (X,τ) is *compact* (or *bicompact*[3]) if from every covering of K by sets that are open in X one can select a finite number of sets that cover K.

Example 1. An interval $[a,b]$ of the set $\mathbb{R}$ of real numbers in the standard topology is a compact set, as follows immediately from the lemma of Subsect. 2.1.3 asserting that one can select a finite covering from any covering of a closed interval by open intervals.

In general an m-dimensional closed interval $I^m = \{x \in \mathbb{R}^m \mid a^i \leq x^i \leq b^i, i = 1,\dots,m\}$ in $\mathbb{R}^m$ is a compact set, as was established in Subsect. 7.1.3.

It was also proved in Subsect. 7.1.3 that a subset of $\mathbb{R}^m$ is compact if and only if it is closed and bounded.

In contrast to the relative properties of being open and closed, the property of compactness is absolute, in the sense that it is independent of the ambient space. More precisely, the following proposition holds.

Proposition 1. *A subset K of a topological space (X,τ) is a compact subset of X if and only if K is compact as a subset of itself with the topology induced from (X,τ).*

Proof. This proposition follows from the definition of compactness and the fact that every set G_K that is open in K can be obtained as the intersection of K with some set G_X that is open in X. $\square$

Thus, if (X,τ_X) and (Y,τ_Y) are two topological spaces that induce the same topology on $K \subset X \cap Y$, then K is simultaneously compact or not compact in both X and Y.

Example 2. Let d be the standard metric on $\mathbb{R}$ and $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$ the unit interval in $\mathbb{R}$. The metric space (I,d) is closed (in itself) and bounded, but is not a compact set, since for example, it is not a compact subset of $\mathbb{R}$.

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[3] The concept of compactness introduced by Definition 1 is sometimes called *bicompactness* in topology.

We now establish the most important properties of compact sets.

Lemma 1. (Compact sets are closed). *If K is a compact set in a Hausdorff space (X, τ), then K is a closed subset of X.*

Proof. By the criterion for a set to be closed, it suffices to verify that every limit point of K, $x_0 \in X$, belongs to K.

Suppose $x_0 \notin K$. For each point $x \in K$ we construct an open neighborhood $G(x)$ such that x_0 has a neighborhood disjoint from $G(x)$. The set $G(x)$, $x \in K$, of all such neighborhoods forms an open covering of K, from which one can select a finite covering $G(x_1), \dots, G(x_n)$. Now if $O_i(x_0)$ is a neighborhood of x_0 such that $G(x_i) \cap O_i(x_0) = \varnothing$, the set $O(x) = \bigcap\limits_{i=1}^{n} O_i(x_0)$ is also a neighborhood of x_0, and $G(x_i) \cap O(x_0) = \varnothing$ for all $i = 1, \dots, n$. But this means that $K \cap O(x_0) = \varnothing$, and then x_0 cannot be a limit point for K. $\square$

Lemma 2. (Nested compact sets.) *If $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap\limits_{i=1}^{\infty} K_i$ is nonempty.*

Proof. By Lemma 1 the sets $G_i = K_1 \setminus K_i$, $i = 1, \dots, n, \dots$ are open in K_1. If the intersection $\bigcap\limits_{i=1}^{\infty} K_i$ is empty, then the sequence $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ forms a covering of K_1. Extracting a finite covering from it, we find that some element G_m of the sequence forms a covering of K_1. But by hypothesis $K_m = K_1 \setminus G_m \neq \varnothing$. This contradiction completes the proof of Lemma 2. $\square$

Lemma 3. (Closed subsets of compact sets.) *A closed subset F of a compact set K is itself compact.*

Proof. Let $\{G_\alpha, \alpha \in A\}$ be an open covering of F. Adjoining to this collection the open set $G = K \setminus F$, we obtain an open covering of the entire compact set K. From this covering we can extract a finite covering of K. Since $G \cap F = \varnothing$, it follows that the set $\{G_\alpha, \alpha \in A\}$ contains a finite covering of F. $\square$

9.3.2 Metric Compact Sets

We shall establish below some properties of metric compact sets, that is, metric spaces that are compact sets with respect to the topology induced by the metric.

Definition 2. The set $E \subset X$ is called an *ε-grid* in the metric space (X, d) if for every point $x \in X$ there is a point $e \in E$ such that $d(e, x) < \varepsilon$.

Lemma 4. (Finite ε-grids.) *If a metric space (K, d) is compact, then for every $\varepsilon > 0$ there exists a finite ε-grid in X.*

Proof. For each point $x \in K$ we choose an open ball $B(x, \varepsilon)$. From the open covering of K by these balls we select a finite covering $B(x_1, \varepsilon), \dots, B(x_n, \varepsilon)$. The points $x_1, \dots, x_n$ obviously form the required ε-grid. □

In analysis, besides arguments that involve the extraction of a finite covering, one often encounters arguments in which a convergent subsequence is extracted from an arbitrary sequence. As it happens, the following proposition holds.

Proposition 2. (Criterion for compactness in a metric space.) *A metric space (K, d) is compact if and only if from each sequence of its points one can extract a subsequence that converges to a point of K.*

The convergence of the sequence $\{x_n\}$ to some point $a \in K$, as before, means that for every neighborhood $U(a)$ of the point $a \in K$ there exists an index $N \in \mathbb{N}$ such that $x_n \in U(a)$ for $n > N$.

We shall discuss the concept of limit in more detail below in Sect. 9.6.

We preface the proof of Proposition 2 with two lemmas.

Lemma 5. *If a metric space (K, d) is such that from each sequence of its points one can select a subsequence that converges in K, then for every $\varepsilon > 0$ there exists a finite ε-grid.*

Proof. If there were no finite ε_0-grid for some $\varepsilon_0 > 0$, one could construct a sequence $\{x_n\}$ of points in K such that $d(x_n, x_i) > \varepsilon_0$ for all $n \in \mathbb{N}$ and all $i \in \{1, \dots, n-1\}$. Obviously it is impossible to extract a convergent subsequence of this sequence. □

Lemma 6. *If the metric space (K, d) is such that from each sequence of its points one can select a subsequence that converges in K, then every nested sequence of nonempty closed subsets of the space has a nonempty intersection.*

Proof. If $F_1 \supset \dots \supset F_n \supset \cdots$ is the sequence of closed sets, then choosing one point of each, we obtain a sequence $x_1, \dots, x_n, \dots$, from which we extract a convergent subsequence $\{x_{n_i}\}$. The limit $a \in K$ of this sequence, by construction, necessarily belongs to each of the closed sets F_i, $i \in \mathbb{N}$. □

We can now prove Proposition 2.

Proof. We first verify that if (K, d) is compact and $\{x_n\}$ a sequence of points in it, one can extract a subsequence that converges to some point of K. If the sequence $\{x_n\}$ has only a finite number of different values, the assertion is obvious. Therefore we may assume that the sequence $\{x_n\}$ has infinitely many different values. For $\varepsilon_1 = 1/1$, we construct a finite 1-grid and take a closed ball $\widetilde{B}(a_1, 1)$ that contains an infinite number of terms of the sequence. By Lemma 3 the ball $\widetilde{B}(a_1, 1)$ is itself a compact set, in which there exists a finite $\varepsilon_2 = 1/2$-grid and a ball $\widetilde{B}(a_2, 1/2)$ containing infinitaly many elements of the sequence. In this way a nested sequence of compact sets $\widetilde{B}(a_1, 1) \supset \widetilde{B}(a_2, 1/2) \supset \cdots \supset \widetilde{B}(a_n, 1/n) \supset \cdots$ arises, and by Lemma 2 has a common point $a \in K$. Choosing a point x_{n_1} of the sequence $\{x_n\}$ in the ball $\widetilde{B}(a_1, 1)$, then a point x_{n_2} in $\widetilde{B}(a_2, 1/2)$ with $n_2 > n_1$, and so on, we obtain a subsequence $\{x_{n_i}\}$ that converges to a by construction.

We now prove the converse, that is, we verify that if from every sequence $\{x_n\}$ of points of the metric space (K, d) one can select a subsequence that converges in K, then (K, d) is compact.

In fact, if there is some open covering $\{G_\alpha, \alpha \in A\}$ of the space (K, d) from which one cannot select a finite covering, then using Lemma 5 to construct a finite 1-grid in K, we find a closed ball $\widetilde{B}(a_1, 1)$, that also cannot be covered by a finite collection of sets of the system $\{B_\alpha, \alpha \in A\}$.

The ball $\widetilde{B}(a_1, 1)$ can now be regarded as the initial set, and, constructing a finite 1/2-grid in it, we find in it a ball $\widetilde{B}(a_2, 1/2)$ that does not admit covering by a finite number of sets in the system $\{G_\alpha, \alpha \in A\}$.

The resulting nested sequence of closed sets $\widetilde{B}(a_1, 1) \supset \widetilde{B}(a_2, 1/2) \supset \cdots \supset \widetilde{B}(a_n, 1/n) \supset \cdots$ has a common point $a \in K$ by Lemma 6, and the construction shows that there is only one such point. This point is covered by some set G_{α_0} of the system; and since G_{α_0} is open, all the sets $\widetilde{B}(a_n, 1/n)$ must be contained in G_{α_0} for sufficiently large values of n. This contradiction completes the proof of the proposition. □

9.3.3 Problems and Exercises

1. A subset of a metric space is *totally bounded* if for every $\varepsilon > 0$ it has a finite ε-grid.

a) Verify that total boundedness of a set is unaffected, whether one forms the grid from points of the set itself or from points of the ambient space.

b) Show that a subset of a metric space is compact if and only if it is totally bounded and closed.

c) Show by example that a closed bounded subset of a metric space is not always totally bounded, and hence not always compact.

2. A subset of a topological space is *relatively (or conditionally) compact* if its closure is compact.

Give examples of relatively compact subsets of $\mathbb{R}^n$.

3. A topological space is *locally compact* if each point of the space has a relatively compact neighborhood.

Give examples of locally compact topological spaces that are not compact.

4. Show that for every locally compact, but not compact topological space (X, τ_X) there is a compact topological space (Y, τ_Y) such that $X \subset Y$, $Y \setminus X$ consists of a single point, and the space (X, τ_X) is a subspace of the space (Y, τ_Y).

9.4 Connected Topological Spaces

Definition 1. A topological space (X, τ) is *connected* if it contains no open-closed sets[4] except X itself and the empty set.

This definition will become more transparent to intuition if we recast it in the following form.

A topological space is connected if and only if it cannot be represented as the union of two disjoint nonempty closed sets (or two disjoint nonempty open sets).

Definition 2. A set E in a topological space (X, τ) is *connected* if it is connected as a topological subspace of (X, τ) (with the induced topology).

It follows from this definition and Definition 1 that the property of a set of being connected is independent of the ambient space. More precisely, if (X, τ_X) and (Y, τ_Y) are topological spaces containing E and inducing the same topology on E, then E is connected or not connected simultaneously in both X and Y.

Example 1. Let $E = \{x \in \mathbb{R} | x \neq 0\}$. The set $E_- = \{x \in E | x < 0\}$ is nonempty, not equal to E, and at the same time open-closed in E (as is $E_+ = \{x \in \mathbb{R} | x > 0\}$), if E is regarded as a topological space with the topology induced by the standard topology of $\mathbb{R}$. Thus, as our intuition suggests, E is not connected.

Proposition. (Connected subsets of $\mathbb{R}$.) *A nonempty set $E \subset \mathbb{R}$ is connected if and only if for any x and z belonging to E, the inequalities $x < y < z$ imply that $y \in E$.*

Thus, the only connected subsets of the line are intervals (finite or infinite): open, half-open, and closed.

Proof. Necessity. Let E be a connected subset of $\mathbb{R}$, and let the triple of points a, b, c be such that $a \in E$, $b \in E$, but $c \notin E$, even though $a < c < b$. Setting $A = \{x \in E | x < c\}$, $B = \{x \in E | x > c\}$, we see that $a \in A$, $b \in B$, that is, $A \neq \varnothing$, $B \neq \varnothing$, and $A \cap B = \varnothing$. Moreover $E = A \cup B$, and both sets A and B are open in E. This contradicts the connectedness of E.

Sufficiency. Let E be a subspace of $\mathbb{R}$ having the property that together with any pair of points a and b belonging to it, every point between them in the closed interval $[a, b]$ also belongs to E. We shall show that E is connected.

Suppose that A is an open-closed subset of E with $A \neq \varnothing$ and $B = E \setminus A \neq \varnothing$. Let $a \in A$ and $b \in B$. For definiteness we shall assume that $a < b$. (We certainly have $a \neq b$, since $A \cap B = \varnothing$.) Consider the point

[4] That is, sets that are simultaneously open and closed.

$c_1 = \sup\{A \cap [a, b]\}$. Since $A \ni a \le c_1 \le b \in B$, we have $c_1 \in E$. Since A is closed in E, we conclude that $c_1 \in A$.

Considering now the point $c_2 = \inf\{B \cap [c_1, b]\}$ we conclude similarly, since B is closed, that $c_2 \in B$. Thus $a \le c_1 < c_2 \le b$, since $c_1 \in A$, $c_2 \in B$, and $A \cap B = \varnothing$. But it now follows from the definition of c_1 and c_2 and the relation $E = A \cup B$ that no point of the open interval $]c_1, c_2[$ can belong to E. This contradicts the original property of E. Thus the set E cannot have a subset A with these properties, and that proves that E is connected. □

9.4.1 Problems and Exercises

1. a) Verify that if A is an open-closed subset of (X, τ), then $B = X \setminus A$ is also such a set.

b) Show that in terms of the ambient space the property of connectedness of a set can be expressed as follows: *A subset E of a topological space (X, τ) is connected if and only if there is no pair of open (or closed) subsets G'_X, G''_X that are disjoint and such that $E \cap G'_X \ne \varnothing$, $E \cap G''_X \ne \varnothing$, and $E \subset G'_X \cup G''_X$.*

2. Show the following:

a) The union of connected subspaces having a common point is connected.

b) The intersection of connected subspaces is not always connected.

c) The closure of a connected subspace is connected.

3. One can regard the group $GL(n)$ of nonsingular $n \times n$ matrices with real entries as an open subset in the product space $\mathbb{R}^{n^2}$, if each element of the matrix is associated with a copy of the set $\mathbb{R}$ of real numbers. Is the space $GL(n)$ connected?

4. A topological space is *locally connected* if each of its points has a connected neighborhood.

a) Show that a locally connected space may fail to be connected.

b) The set E in $\mathbb{R}^2$ consists of the graph of the function $x \mapsto \sin \frac{1}{x}$ (for $x \ne 0$) plus the closed interval $\{(x, y) \in \mathbb{R}^2 | \, x = 0 \wedge |y| \le 1\}$ on the y-axis. The set E is endowed with the topology induced from $\mathbb{R}^2$. Show that the resulting topological space is connected but not locally connected.

5. In Subsect. 7.2.2 we defined a connected subset of $\mathbb{R}^n$ as a set $E \subset \mathbb{R}^n$ any two of whose points can be joined by a path whose support lies in E. In contrast to the definition of topological connectedness introduced in the present section, the concept we considered in Chapt. 7 is usually called *path connectedness* or *arcwise connectedness*. Verify the following:

a) A path-connected subset of $\mathbb{R}^n$ is connected.

b) Not every connected subset of $\mathbb{R}^n$ with $n > 1$ is path connected. (See Problem 4.)

c) Every connected open subset of $\mathbb{R}^n$ is path connected.

9.5 Complete Metric Spaces

In this section we shall be discussing only metric spaces, more precisely, a class of such spaces that plays an important role in various areas of analysis.

9.5.1 Basic Definitions and Examples

By analogy with the concepts that we already know from our study of the space $\mathbb{R}^n$, we introduce the concepts of fundamental (Cauchy) sequences and convergent sequences of points of an arbitrary metric space.

Definition 1. A sequence $\{x_n; n \in \mathbb{N}\}$ of points of a metric space (X, d) is a *fundamental* or *Cauchy* sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for any indices $m, n \in \mathbb{N}$ larger than N.

Definition 2. A sequence $\{x_n; n \in \mathbb{N}\}$ of points of a metric space (X, d) *converges to the point* $a \in X$ and a is its *limit* if $\lim\limits_{n\to\infty} d(a, x_n) = 0$.

A sequence that has a limit will be called *convergent*, as before.

We now give the basic definition.

Definition 3. A metric space (X, d) is *complete* if every Cauchy sequence of its points is convergent.

Example 1. The set $\mathbb{R}$ of real numbers with the standard metric is a complete metric space, as follows from the Cauchy criterion for convergence of a numerical sequence.

We remark that, since every convergent sequence of points in a metric space is obviously a Cauchy sequence, the definition of a complete metric space essentially amounts to simply postulating the Cauchy convergence criterion for it.

Example 2. If the number 0, for example, is removed from the set $\mathbb{R}$, the remaining set $\mathbb{R} \setminus 0$ will not be a complete space in the standard metric. Indeed, the sequence $x_n = 1/n$, $n \in \mathbb{N}$, is a Cauchy sequence of points of this set, but has no limit in $\mathbb{R} \setminus 0$.

Example 3. The space $\mathbb{R}^n$ with any of its standard metrics is complete, as was explained in Subsect. 7.2.1.

Example 4. Consider the set $C[a, b]$ of real-valued continuous functions on a closed interval $[a, b] \subset \mathbb{R}$, with the metric

$$d(f, g) = \max_{a \le x \le b} |f(x) - g(x)| \tag{9.9}$$

(see Sect. 9.1, Example 7). We shall show that the metric space $C[a, b]$ is complete.

Proof. Let $\{f_n(x) : n \in \mathbb{N}\}$ be a Cauchy sequence of functions in $C[a,b]$, that is,

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \;\; \forall m \in \mathbb{N} \;\; \forall n \in \mathbb{N} \left((m > N \wedge n > N) \Longrightarrow \right.$$
$$\left. \Longrightarrow \quad \forall x \in [a,b] \left(|f_m(x) - f_n(x)| < \varepsilon \right) \right). \quad (9.10)$$

For each fixed value of $x \in [a,b]$, as one can see from (9.10), the numerical sequence $\{f_n(x); n \in \mathbb{N}\}$ is a Cauchy sequence and hence has a limit $f(x)$ by the Cauchy convergence criterion.

Thus

$$f(x) := \lim_{n\to\infty} f_n(x) \,, \quad x \in [a,b] \,. \quad (9.11)$$

We shall verify that the function $f(x)$ is continuous on $[a,b]$, that is, $f \in C[a,b]$.

It follows from (9.10) and (9.11) that the inequality

$$|f(x) - f_n(x)| \le \varepsilon \quad \forall x \in [a,b] \quad (9.12)$$

holds for $n > N$.

We fix the point $x \in [a,b]$ and verify that the function f is continuous at this point. Suppose the increment h is such that $(x+h) \in [a,b]$. The identity

$$f(x+h) - f(x) = f(x+h) - f_n(x+h) + f_n(x+h) - f_n(x) + f_n(x) - f(x)$$

implies the inequality

$$|f(x+h) - f(x)| \le |f(x+h) - f_n(x+h)|$$
$$+|f_n(x+h) - f_n(x)| + |f_n(x) - f(x)|. \quad (9.13)$$

By virtue of (9.12) the first and last terms on the right-hand side of this last inequality do not exceed ε if $n > N$. Fixing $n > N$, we obtain a function $f_n \in C[a,b]$, and then choosing $\delta = \delta(\varepsilon)$ such that $|f_n(x+h) - f_n(x)| < \varepsilon$ for $|h| < \delta$, we find that $|f(x+h) - f(x)| < 3\varepsilon$ if $|h| < \delta$. But this means that the function f is continuous at the point x. Since x was an arbitrary point of the closed interval $[a,b]$, we have shown that $f \in C[a,b]$. □

Thus the space $C[a,b]$ with the metric (9.9) is a complete metric space. This is a very important fact, one that is widely used in analysis.

Example 5. If instead of the metric (9.9) we consider the integral metric

$$d(f,g) = \int_a^b |f - g|(x)\, dx \quad (9.14)$$

on the same set $C[a,b]$, the resulting metric space is no longer complete.

Proof. For the sake of notational simplicity, we shall assume $[a, b] = [-1, 1]$ and consider, for example, the sequence $\{f_n \in C[-1, 1];\ n \in \mathbb{N}\}$ of functions defined as follows:

$$f_n(x) = \begin{cases} -1, & \text{if} \quad -1 \le x \le -1/n\,, \\ nx, & \text{if} \quad -1/n < x < 1/n\,, \\ 1, & \text{if} \quad 1/n \le x \le 1\,. \end{cases}$$

(See Fig. 9.2.)

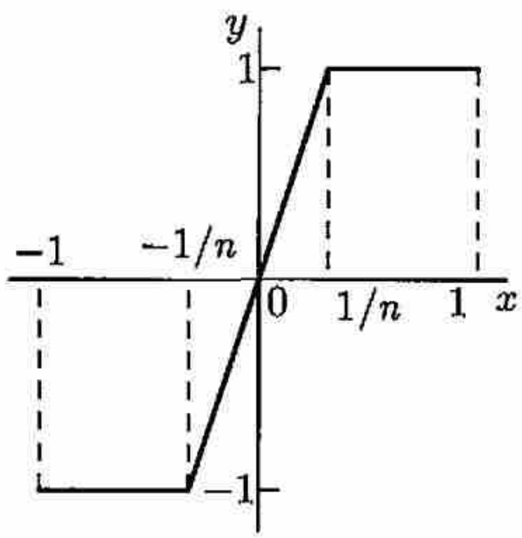

Fig. 9.2.

It follows immediately from properties of the integral that this sequence is a Cauchy sequence in the sense of the metric (9.14) in $C[-1, 1]$. At the same time, it has no limit in $C[-1, 1]$. For if a continuous function $f \in C[-1, 1]$ were the limit of this sequence in the sense of metric (9.14), then f would have to be constant on the interval $-1 \le x < 0$ and equal to -1 while at the same time it would have to be constant and equal to 1 on the interval $0 < x \le 1$, which is incompatible with the continuity of f at the point $x = 0$. □

Example 6. It is slightly more difficult to show that even the set $\mathcal{R}[a, b]$ of real-valued Riemann-integrable functions defined on the closed interval $[a, b]$ is not complete in the sense of the metric 9.14.[5] We shall show this, using the Lebesgue criterion for Riemann integrability of a function.

Proof. We take $[a, b]$ to be the closed interval $[0, 1]$, and we shall construct a Cantor set on it that is not a set of measure zero. Let $\Delta \in]0, 1/3[$. We remove from the interval $[0, 1]$ the middle piece of it of length Δ. More precisely, we remove the $\Delta/2$-neighborhood of the midpoint of the closed interval $[0, 1]$.

[5] In regard to the metric (9.14) on $\mathcal{R}[a, b]$ see the remark to Example 9 in Sect. 9.1.

On each of the two remaining intervals, we remove the middle piece of length $\Delta \cdot 1/3$. On each of the four remaining closed intervals we remove the middle piece of length $\Delta \cdot 1/3^2$, and so forth. The length of the intervals removed in this process is $\Delta + \Delta \cdot 2/3 + \Delta \cdot 4/3^2 + \cdots + \Delta \cdot (2/3)^n + \cdots = 3\Delta$. Since $0 < \Delta < 1/3$, we have $1 - 3\Delta > 0$, and, as one can verify, it follows from this that the (Cantor) set K remaining on the closed interval $[0, 1]$ does not have measure zero in the sense of Lebesgue.

Now consider the following sequence: $\{f_n \in \mathcal{R}[0,1];\ n \in \mathbb{N}\}$. Let f_n be a function equal to 1 everywhere on $[0, 1]$ except at the points of the intervals removed at the first n steps, where it is set equal to zero. It is easy to verify that this sequence is a Cauchy sequence in the sense of the metric (9.14). If some function $f \in \mathcal{R}[0,1]$ were the limit of this sequence, then f would have to be equal to the characteristic function of the set K at almost every point of the interval $[0, 1]$. Then f would have discontinuities at all points of the set K. But, since K does not have measure 0, one could conclude from the Lebesgue criterion that $f \notin \mathcal{R}[0,1]$. Hence $\mathcal{R}[a,b]$ with the metric (9.14) is not a complete metric space. □

9.5.2 The Completion of a Metric Space

Example 7. Let us return again to the real line and consider the set $\mathbb{Q}$ of rational numbers with the metric induced by the standard metric on $\mathbb{R}$.

It is clear that a sequence of rational numbers converging to $\sqrt{2}$ in $\mathbb{R}$ is a Cauchy sequence, but does not have a limit in $\mathbb{Q}$, that is, $\mathbb{Q}$ is not a complete space with this metric. However, $\mathbb{Q}$ happens to be a subspace of the complete metric space $\mathbb{R}$, which it is natural to regard as the completion of $\mathbb{Q}$. Note that the set $\mathbb{Q} \subset \mathbb{R}$ could also be regarded as a subset of the complete metric space $\mathbb{R}^2$, but it does not seem reasonable to call $\mathbb{R}^2$ the completion of $\mathbb{Q}$.

Definition 4. The smallest complete metric space containing a given metric space (X, d) is the *completion* of (X, d).

This intuitively acceptable definition requires at least two clarifications: what is meant by the "smallest" space, and does it exist?

We shall soon be able to answer both of these questions; in the meantime we adopt the following more formal definition.

Definition 5. If a metric space (X, d) is a subspace of the metric space (Y, d) and the set $X \subset Y$ is everywhere dense in Y, the space (Y, d) is called a *completion* of the metric space (X, d).

Definition 6. We say that the metric space (X_1, d_1) is *isometric* to the metric space (X_2, d_2) if there exists a bijective mapping $f : X_1 \to X_2$ such that $d_2\big(f(a), f(b)\big) = d_1(a, b)$ for any points a and b in X_1. (The mapping $f : X_1 \to X_2$ is called an *isometry* in that case.)

It is clear that this relation is reflexive, symmetric, and transitive, that is, it is an equivalence relation between metric spaces. In studying the properties of metric spaces we study not the individual space, but the properties of all spaces isometric to it. For that reason one may regard isometric spaces as identical.

Example 8. Two congruent figures in the plane are isometric as metric spaces, so that in studying the metric properties of figures we abstract completely, for example, from the location of a figure in the plane, identifying all congruent figures.

By adopting the convention of identifying isometric spaces, one can show that if the completion of a metric space exists at all, it is unique.

As a preliminary, we verify the following statement.

Lemma. *The following inequality holds for any quadruple of points a, b, u, v of the metric space (X, d):*

$$|d(a,b) - d(u,v)| \leq d(a,u) + d(b,v) . \tag{9.15}$$

Proof. By the triangle inequality

$$d(a,b) \leq d(a,u) + d(u,v) + d(b,v) .$$

By the symmetry of the points, this relation implies (9.15). □

We now prove uniqueness of the completion.

Proposition 1. *If the metric spaces (Y_1, d_1) and (Y_2, d_2) are completions of the same space (X, d), then they are isometric.*

Proof. We construct an isometry $f : Y_1 \to Y_2$ as follows. For $x \in X$ we set $f(x) = x$. Then $d_2\big(f(x_1), f(x_2)\big) = d\big(f(x_1), f(x_2)\big) = d(x_1, x_2) = d_1(x_1, x_2)$ for $x_1, x_2 \in X$. If $y_1 \in Y_1 \setminus X$, then y_1 is a limit point for X, since X is everywhere dense in Y_1. Let $\{x_n;\, n \in \mathbb{N}\}$ be a sequence of points of X converging to y_1 in the sense of the metric d_1. This sequence is a Cauchy sequence in the sense of the metric d_1. But since the metrics d_1 and d_2 are both equal to d on X, this sequence is also a Cauchy sequence in (Y_2, d_2). The latter space is complete, and hence this sequence has a limit $y_2 \in Y_2$. It can be verified in the standard manner that this limit is unique. We now set $f(y_1) = y_2$. Since any point $y_2 \in Y_2 \setminus X$, just like any point $y_1 \in Y_1 \setminus X$, is the limit of a Cauchy sequence of points in X, the mapping $f : Y_1 \to Y_2$ so constructed is surjective.

We now verify that

$$d_2\big(f(y_1'), f(y_1'')\big) = d_1(y_1', y_1'') \tag{9.16}$$

for any pair of points y_1', y_1'' of Y_1.

If y_1' and y_1'' belong to X, this equality is obvious. In the general case we take two sequences $\{x_n'; n \in \mathbb{N}\}$ and $\{x_n''; n \in \mathbb{N}\}$ converging to y_1' and y_1'' respectively. It follows from inequality (9.15) that

$$d_1(y_1', y_1'') = \lim_{n\to\infty} d_1(x_n', x_n'') ,$$

or, what is the same,

$$d_1(y_1', y_1'') = \lim_{n\to\infty} d(x_n', x_n'') . \qquad (9.17)$$

By construction these same sequences converge to $y_2' = f(y_1')$ and $y_2'' = f(y_2'')$ respectively in the space (Y_2, d_2). Hence

$$d_2(y_2', y_2'') = \lim_{n\to\infty} d(x_n', x_n'') . \qquad (9.18)$$

Comparing relations (9.17) and (9.18), we obtain Eq. (9.16). This equality then simultaneously establishes that the mapping $f : Y_1 \to Y_2$ is injective and hence completes the proof that f is an isometry. $\square$

In Definition 5 of the completion (Y, d) of a metric space (X, d) we required that (X, d) be a subspace of (Y, d) that is everywhere dense in (Y, d). Under the identification of isometric spaces one could now broaden the idea of a completion and adopt the following definition.

Definition 5′. A metric space (Y, d_Y) is a *completion* of the metric space (X, d_X) if there is a dense subspace of (Y, d_Y) isometric to (X, d_X).

We now prove the existence of a completion.

Proposition 2. *Every metric space has a completion.*

Proof. If the initial space itself is complete, then it is its own completion.

We have already essentially demonstrated the idea for constructing the completion of an incomplete metric space (X, d_X) when we proved Proposition 1.

Consider the set of Cauchy sequences in the space (X, d_X). Two such sequences $\{x_n'; n \in \mathbb{N}\}$ and $\{x_n''; n \in \mathbb{N}\}$ are called *equivalent* or *confinal* if $d_X(x_n', x_n'') \to 0$ as $n \to \infty$. It is easy to see that confinality really is an equivalence relation. We shall denote the set of equivalence classes of Cauchy sequences by S. We introduce a metric in S by the following rule. If s' and s'' are elements of S, and $\{x_n'; n \in \mathbb{N}\}$ and $\{x_n''; n \in \mathbb{N}\}$ are sequences from the classes s' and s'' respectively, we set

$$d(s', s'') = \lim_{n\to\infty} d_X(x_n', x_n'') . \qquad (9.19)$$

It follows from inequality (9.15) that this definition is unambiguous: the limit written on the right exists (by the Cauchy criterion for a numerical sequence) and is independent of the choice of the individual sequences $\{x_n'; n \in \mathbb{N}\}$ and $\{x_n''; n \in \mathbb{N}\}$ from the classes s' and s''

The function $d(s', s'')$ satisfies all the axioms of a metric. The resulting metric space (S, d) is the required completion of the space (X, d_X). Indeed, (X, d_X) is isometric to the subspace (S_X, d) of the space (S, d) consisting of the equivalence classes of fundamental sequences that contain constant sequences $\{x_n = x \in X;\ n \in \mathbb{N}\}$. It is natural to identify such a class $s \in S$ with the point $x \in X$. The mapping $f : (X, d_X) \to (S_X, d)$ is obviously an isometry.

It remains to be verified that (S_X, d) is everywhere dense in (S, d) and that (S, d) is a complete metric space.

We first verify that (S_X, d) is dense in (S, d). Let s be an arbitrary element of S and $\{x_n;\ n \in \mathbb{N}\}$ a Cauchy sequence in (X, d_X) belonging to the class $s \in S$. Taking $\xi_n = f(x_n)$, $n \in \mathbb{N}$, we obtain a sequence $\{\xi_n;\ n \in \mathbb{N}\}$ of points of (S_X, d) that has precisely the element $s \in S$ as its limit, as one can see from (9.19).

We now prove that the space (S, d) is complete. Let $\{s_n;\ n \in \mathbb{N}\}$ be an arbitrary Cauchy sequence in the space (S, d). For each $n \in \mathbb{N}$ we choose an element ξ_n in (S_X, d) such that $d(s_n, \xi_n) < 1/n$. Then the sequence $\{\xi_n;\ n \in \mathbb{N}\}$, like the sequence $\{s_n;\ n \in \mathbb{N}\}$, is a Cauchy sequence. But in that case the sequence $\{x_n = f^{-1}(\xi_n);\ n \in \mathbb{N}\}$ will also be a Cauchy sequence. The sequence $\{x_n;\ n \in \mathbb{N}\}$ defines an element $s \in S$, to which the given sequence $\{s_n;\ n \in \mathbb{N}\}$ converges by virtue of relation (9.19). □

Remark 1. Now that Propositions 1 and 2 have been proved, it becomes understandable that the completion of a metric space in the sense of Definition 5′ is indeed the smallest complete space containing (up to isometry) the given metric space. In this way we have justified the original Definition 4 and made it precise.

Remark 2. The construction of the set $\mathbb{R}$ of real numbers, starting from the set $\mathbb{Q}$ of rational numbers could have been carried out exactly as in the construction of the completion of a metric space, which was done in full generality above. That is exactly how the transition from $\mathbb{Q}$ to $\mathbb{R}$ was carried out by Cantor.

Remark 3. In Example 6 we showed that the space $\mathcal{R}[a, b]$ of Riemann-integrable functions is not complete in the natural integral metric. Its completion is the important space $\mathcal{L}[a, b]$ of Lebesgue-integrable functions.

9.5.3 Problems and Exercises

1. a) Prove the following n e s t e d b a l l l e m m a. *Let (X, d) be a metric space and $\widetilde{B}(x_1, r_1) \supset \cdots \supset \widetilde{B}(x_n, r_n) \supset \cdots$ a nested sequence of closed balls in X whose radii tend to zero. The space (X, d) is complete if and only if for every such sequence there exists a unique point belonging to all the balls of the sequence.*

b) Show that if the condition $r_n \to 0$ as $n \to \infty$ is omitted from the lemma stated above, the intersection of a nested sequence of balls may be empty, even in a complete space.

2. a) A set $E \subset X$ of a metric space (X, d) is *nowhere dense* in X if it is not dense in any ball, that is, if for every ball $B(x, r)$ there is a second ball $B(x_1, r_1) \subset B(x, r)$ containing no points of the set E.

A set E is *of first category* in X if it can be represented as a countable union of nowhere dense sets.

A set that is not of first category is *of second category* in X.

Show that a complete metric space is a set of second category (in itself).

b) Show that if a function $f \in C^{(\infty)}[a, b]$ is such that $\forall x \in [a, b]\ \exists n \in \mathbb{N}\ \forall m > n\ (f^{(m)}(x) = 0)$, then the function f is a polynomial.

9.6 Continuous Mappings of Topological Spaces

From the point of view of analysis, the present section and the one following contain the most important results in the present chapter.

The basic concepts and propositions discussed here form a natural, sometimes verbatim extension to the case of mappings of arbitrary topological or metric spaces, of concepts and propositions that are already well known to us in . In the process, not only the statement but also the proofs of many facts turn out to be identical with those already considered; in such cases the proofs are naturally omitted with a reference to the corresponding propositions that were discussed in detail earlier.

9.6.1 The Limit of a Mapping

a. The Basic Definition and Special Cases of it

Definition 1. Let $f : X \to Y$ be a mapping of the set X with a fixed base $\mathcal{B} = \{B\}$ in X into a topological space Y. The point $A \in Y$ is the *limit of the mapping $f : X \to Y$ over the base $\mathcal{B}$*, and we write $\lim\limits_{\mathcal{B}} f(x) = A$, if for every neighborhood $V(A)$ of A in Y there exists an element $B \in \mathcal{B}$ of the base $\mathcal{B}$ whose image under the mapping f is contained in $V(A)$.

In logical symbols Definition 1 has the form

$$\lim_{\mathcal{B}} f(x) = A := \forall V(A) \subset Y\ \exists B \in \mathcal{B}\ \big(f(B) \subset V(A)\big)\ .$$

We shall most often encounter the case in which X, like Y, is a topological space and $\mathcal{B}$ is the base of neighborhoods or deleted neighborhoods of some point $a \in X$. Retaining our earlier notation $x \to a$ for the base of deleted neighborhoods $\{\overset{\circ}{U}(a)\}$ of the point a, we can specialize Definition 1 for this base:

$$\lim_{x \to a} f(x) = A := \forall V(A) \subset Y\ \exists \overset{\circ}{U}(a) \subset X\ \big(f\big(\overset{\circ}{U}(a)\big) \subset V(A)\big)\ .$$

If (X, d_X) and (Y, d_Y) are metric spaces, this last definition can be restated in ε-δ language:

$$\lim_{x \to a} f(x) = A := \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X$$
$$\big(0 < d_X(a, x) < \delta \Longrightarrow d_Y\big(A, f(x)\big) < \varepsilon\big) .$$

In other words,

$$\lim_{x \to a} f(x) = A \Longleftrightarrow \lim_{x \to a} d_Y\big(A, f(x)\big) = 0 .$$

Thus we see that, having the concept of a neighborhood, one can define the concept of the limit of a mapping $f : X \to Y$ into a topological or metric space Y just as was done in the case $Y = \mathbb{R}$ or, more generally, $Y = \mathbb{R}^n$.

b. Properties of the Limit of a Mapping We now make some remarks on the general properties of the limit.

We first note that the uniqueness of the limit obtained earlier no longer holds when Y is not a Hausdorff space. But if Y is a Hausdorff space, then the limit is unique and the proof does not differ at all from the one given in the special cases $Y = \mathbb{R}$ or $Y = \mathbb{R}^n$.

Next, if $f : X \to Y$ is a mapping into a metric space, it makes sense to speak of the *boundedness* of the mapping (meaning the boundedness of the set $f(X)$ in Y), and of *ultimate boundedness* of a mapping with respect to the base $\mathcal{B}$ in X (meaning that there exists an element B of $\mathcal{B}$ on which f is bounded).

It follows from the definition of a limit that if a mapping $f : X \to Y$ of a set X with base $\mathcal{B}$ into a metric space Y has a limit over the base $\mathcal{B}$, then it is ultimately bounded over that base.

c. Questions Involving the Existence of the Limit of a Mapping

Proposition 1. (Limit of a composition of mappings.) *Let Y be a set with base $\mathcal{B}_Y$ and $g : Y \to Z$ a mapping of Y into a topological space Z having a limit over the base $\mathcal{B}_Y$.*

Let X be a set with base $\mathcal{B}_X$ and $f : X \to Y$ a mapping of X into Y such that for every element $B_Y \in \mathcal{B}_Y$ there exists an element $B_X \in \mathcal{B}_X$ whose image is contained in B_Y, that is, $f(B_X) \subset B_Y$.

Under these hypotheses the composition $g \circ f : X \to Z$ of the mappings f and g is defined and has a limit over the base $\mathcal{B}_X$, and

$$\lim_{\mathcal{B}_X} g \circ f(x) = \lim_{\mathcal{B}_Y} g(y) .$$

For the proof see Theorem 5 of Sect. 3.2.

Another important proposition on the existence of the limit is the Cauchy criterion, to which we now turn. This time we will be discussing a mapping $f : X \to Y$ into a metric space, and in fact a complete metric space.

In the case of a mapping $f : X \to Y$ of the set X into a metric space (Y, d) it is natural to adopt the following definition.

Definition 2. The *oscillation* of the mapping $f : X \to Y$ on a set $E \subset X$ is the quantity

$$\omega(f, E) = \sup_{x_1, x_2 \in E} d\big(f(x_1), f(x_2)\big) .$$

The following proposition holds.

Proposition 2. (Cauchy criterion for existence of the limit of a mapping.) *Let X be a set with a base $\mathcal{B}$, and let $f : X \to Y$ be a mapping of X into a complete metric space (Y, d).*

A necessary and sufficient condition for the mapping f to have a limit over the base $\mathcal{B}$ is that for every $\varepsilon > 0$ there exist an element B in $\mathcal{B}$ on which the oscillation of the mapping is less than ε.

More briefiy:

$$\exists \lim_{\mathcal{B}} f(x) \Longleftrightarrow \forall \varepsilon > 0 \, \exists B \in \mathcal{B} \, \big(\omega(f, B) < \varepsilon\big) .$$

For the proof see Theorem 4 of Sect. 3.2.

It is useful to remark that the completeness of the space Y is needed only in the implication from the right-hand side to the left-hand side. Moreover, if Y is not a complete space, it is usually this implication that breaks down.

9.6.2 Continuous Mappings

a. Basic Definitions

Definition 3. A mapping $f : X \to Y$ of a topological space (X, τ_X) into a topological space (Y, τ_Y) is *continuous at a point* $a \in X$ if for every neighborhood $V\big(f(a)\big) \subset Y$ of the point $f(a) \in Y$ there exists a neighborhood $U(a) \subset X$ of the point $a \in X$ whose image $f\big(U(a)\big)$ is contained in $V\big(f(a)\big)$.

Thus,

$$\begin{aligned} &f : X \to Y \text{ is continuous at } a \in X := \\ &\qquad = \forall V\big(f(a)\big) \, \exists U(a) \, \Big(f\big(U(a)\big) \subset V\big(f(a)\big)\Big) . \end{aligned}$$

In the case when X and Y are metric spaces (X, d_X) and (Y, d_Y), Definition 3 can of course be stated in ε-δ language:

$$\begin{aligned} &f : X \to Y \text{ is continuous at } a \in X := \\ &= \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \, \Big(d_X(a, x) < \delta \Longrightarrow d_Y\big(f(a), f(x)\big) < \varepsilon\Big) . \end{aligned}$$

Definition 4. The mapping $f : X \to Y$ is *continuous* if it is continuous at each point $x \in X$.

The set of continuous mappings from X into Y will be denoted $C(X, Y)$.

Theorem 1. (Criterion for continuity.) *A mapping $f : X \to Y$ of a topological space (X, τ_X) into a topological space (Y, τ_Y) is continuous if and only if the pre-image of every open (resp. closed) subset of Y is open (resp. closed) in X.*

Proof. Since the pre-image of a complement is the complement of the pre-image, it suffices to prove the assertions for open sets.

We first show that if $f \in C(X,Y)$ and $G_Y \in \tau_Y$, then $G_X = f^{-1}(G_Y)$ belongs to τ_X. If $G_X = \varnothing$, it is immediate that the pre-image is open. If $G_X \neq \varnothing$ and $a \in G_X$, then by definition of continuity of the mapping f at the point a, for the neighborhood G_Y of the point $f(a)$ there exists a neighborhood $U_X(a)$ of $a \in X$ such that $f(U_X(a)) \subset G_Y$. Hence $U_X(a) \subset G_X = f^{-1}(G_Y)$. Since $G_X = \bigcup\limits_{a \in G_X} U_X(a)$, we conclude that G_X is open, that is, $G_X \in \tau_X$.

We now prove that if the pre-image of every open set in Y is open in X, then $f \in C(X,Y)$. But, taking any point $a \in X$ and any neighborhood $V_Y(f(a))$ of its image $f(a)$ in Y, we discover that the set $U_X(a) = f^{-1}(V_Y(f(a)))$ is an open neighborhood of $a \in X$, whose image is contained in $V_Y(f(a))$. Consequently we have verified the definition of continuity of the mapping $f : X \to Y$ at an arbitrary point $a \in X$. $\square$

Definition 5. A bijective mapping $f : X \to Y$ of one topological space (X, τ_X) onto another (Y, τ_Y) is a *homeomorphism* if both the mapping itself and the inverse mapping $f^{-1} : Y \to X$ are continuous.

Definition 6. Topological spaces that admit homeomorphisms onto one another are said to be *homeomorphic*.

As Theorem 1 shows, under a homeomorphism $f : X \to Y$ of the topological space (X, τ_X) onto (Y, τ_Y) the systems of open sets τ_X and τ_Y correspond to each other in the sense that $G_X \in \tau_X \Leftrightarrow f(G_X) = G_Y \in \tau_Y$.

Thus, from the point of view of their topological properties homeomorphic spaces are absolutely identical. Consequently, homeomorphism is the same kind of equivalence relation in the set of all topological spaces as, for example, isometry is in the set of metric spaces.

b. Local Properties of Continuous Mappings We now exhibit the local properties of continuous mappings. They follow immediately from the corresponding properties of the limit.

Proposition 3. (Continuity of a composition of continuous mappings.) *Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be topological spaces. If the mapping $g : Y \to Z$ is continuous at a point $b \in Y$ and the mapping $f : X \to Y$ is continuous at a point $a \in X$ for which $f(a) = b$, then the composition of these mappings $g \circ f : X \to Z$ is continuous at $a \in X$.*

This follows from the definition of continuity of a mapping and Proposition 1.

Proposition 4. (Boundedness of a mapping in a neighborhood of a point of continuity.) *If a mapping $f : X \to Y$ of a topological space (X, τ) into a metric space (Y, d) is continuous at a point $a \in X$, then it is bounded in some neighborhood of that point.*

This proposition follows from the ultimate boundedness (over a base) of a mapping that has a limit.

Before stating the next proposition on properties of continuous mappings, we recall that for mappings into $\mathbb{R}$ or $\mathbb{R}^n$ we defined the quantity

$$\omega(f; a) := \lim_{r \to 0} \omega\big(f, B(a, r)\big)$$

to be the *oscillation of f at the point a*. Since both the concept of the oscillation of a mapping on a set and the concept of a ball $B(a, r)$ make sense in any metric space, the definition of the oscillation $\omega(f, a)$ of the mapping f at the point a also makes sense for a mapping $f : X \to Y$ of a metric space (X, d_X) into a metric space (Y, d_Y).

Proposition 5. *A mapping $f : X \to Y$ of a metric space (X, d_X) into a metric space (Y, d_Y) is continuous at the point $a \in X$ if and only if $\omega(f, a) = 0$.*

This proposition follows immediately from the definition of continuity of a mapping at a point.

c. Global Properties of Continuous Mappings We now discuss some of the important global properties of continuous mappings.

Theorem 2. *The image of a compact set under a continuous mapping is compact.*

Proof. Let $f : K \to Y$ be a continuous mapping of the compact space (K, τ_K) into a topological space (Y, τ_Y), and let $\{G_Y^\alpha, \alpha \in A\}$ be a covering of $f(K)$ by sets that are open in Y. By Theorem 1, the sets $\{G_X^\alpha = f^{-1}(G_Y^\alpha), \alpha \in A\}$ form an open covering of K. Extracting a finite covering $G_X^{\alpha_1}, \dots, G_X^{\alpha_n}$, we find a finite covering $G_Y^{\alpha_1}, \dots, G_Y^{\alpha_n}$ of $f(K) \subset Y$. Thus $f(K)$ is compact in Y. □

Corollary. *A continuous real-valued function $f : K \to \mathbb{R}$ on a compact set assumes its maximal value at some point of the compact set (and also its minimal value, at some point).*

Proof. Indeed, $f(K)$ is a compact set in $\mathbb{R}$, that is, it is closed and bounded. This means that $\inf f(K) \in f(K)$ and $\sup f(K) \in f(K)$. □

In particular, if K is a closed interval $[a, b] \subset \mathbb{R}$, we again obtain the classical theorem of Weierstrass.

Cantor's theorem on uniform continuity carries over verbatim to mappings that are continuous on compact sets. Before stating it, we must give a necessary definition.

Definition 7. A mapping $f : X \to Y$ of a metric space (X, d_X) into a metric space (Y, d_Y) is *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the oscillation $\omega(f, E)$ of f on each set $E \subset X$ of diameter less than δ is less than ε.

Theorem 3. (Uniform continuity.) *A continuous mapping $f : K \to Y$ of a compact metric space K into a metric space (Y, d_Y) is uniformly continuous.*

In particular, if K is a closed interval in $\mathbb{R}$ and $Y = \mathbb{R}$, we again have the classical theorem of Cantor, the proof of which given in Subsect. 4.2.2. carries over with almost no changes to this general case.

Let us now consider continuous mappings of connected spaces.

Theorem 4. *The image of a connected topological space under a continuous mapping is connected.*

Proof. Let $f : X \to Y$ be a continuous mapping of a connected topological space (X, τ_X) onto a topological space (Y, τ_Y). Let E_Y be an open-closed subset of Y. By Theorem 1, the pre-image $E_X = f^{-1}(E_Y)$ of the set E_Y is open-closed in X. By the connectedness of X, either $E_X = \varnothing$ or $E_X = X$. But this means that either $E_Y = \varnothing$ or $E_Y = Y = f(X)$. □

Corollary. *If a function $f : X \to \mathbb{R}$ is continuous on a connected topological space (X, τ) and assumes values $f(a) = A \in \mathbb{R}$ and $f(b) = B \in \mathbb{R}$, then for any number C between A and B there exists a point $c \in X$ at which $f(c) = C$.*

Proof. Indeed, by Theorem 4 $f(X)$ is a connected set in $\mathbb{R}$. But the only connected subsets of $\mathbb{R}$ are intervals (see the Proposition in Sect. 9.4). Thus the point C belongs to $f(X)$ along with A and B. □

In particular, if X is a closed interval, we again have the classical intermediate-value theorem for a continuous real-valued function.

9.6.3 Problems and Exercises

1. a) If the mapping $f : X \to Y$ is continuous, will the images of open (or closed) sets in X be open (or closed) in Y?

b) If the image, as well as the inverse image, of an open set under the mapping $f : X \to Y$ is open, does it necessarily follow that f is a homeomorphism?

c) If the mapping $f : X \to Y$ is continuous and bijective, is it necessarily a homeomorphism?

d) Is a mapping satisfying b) and c) simultaneously a homeomorphism?

2. Show the following.

a) Every continuous bijective mapping of a compact space into a Hausdorff space is a homeomorphism.

b) Without the requirement that the range be a Hausdorff space, the preceding statement is in general not true.

3. Determine whether the following subsets of $\mathbb{R}^n$ are (pairwise) homeomorphic as topological spaces: a line, an open interval on the line, a closed interval on the line; a sphere; a torus.

4. A topological space (X, τ) is *arcwise connected* or *path connected* if any two of its points can be joined by a path lying in X. More precisely, this means that for any points A and B in X there exists a continuous mapping $f : I \to X$ of a closed interval $[a, b] \subset \mathbb{R}$ into X such that $f(a) = A$ and $f(b) = B$.

a) Show that every path connected space is connected.

b) Show that every convex set in $\mathbb{R}^n$ is path connected.

c) Verify that every connected open subset of $\mathbb{R}^n$ is path connected.

d) Show that a sphere $S(a, r)$ is path connected in $\mathbb{R}^n$, but that it may fail to be connected in another metric space, endowed with a completely different topology.

e) Verify that in a topological space it is impossible to join an interior point of a set to an exterior point without intersecting the boundary of the set.

9.7 The Contraction Mapping Principle

Here we shall establish a principle that, despite its simplicity, turns out to be an effective way of proving many existence theorems.

Definition 1. A point $a \in X$ is a *fixed point* of a mapping $f : X \to X$ if $f(a) = a$.

Definition 2. A mapping $f : X \to X$ of a metric space (X, d) into itself is called a *contraction* if there exists a number q, $0 < q < 1$, such that the inequality

$$d\big(f(x_1), f(x_2)\big) \le q d(x_1, x_2) \tag{9.20}$$

holds for any points x_1 and x_2 in X.

Theorem. (Picard[6]–Banach[7] fixed-point principle.) *A contraction mapping $f : X \to X$ of a complete metric space (X, d) into itself has a unique fixed point a.*

Moreover, for any point $x_0 \in X$ the recursively defined sequence x_0, $x_1 = f(x_0)$, …, $x_{n+1} = f(x_n)$,… converges to a. The rate of convergence is given by the estimate

$$d(a, x_n) \le \frac{q^n}{1-q} d(x_1, x_0) . \tag{9.21}$$

---

[6] Ch. É. Picard (1856–1941) – French mathematician who obtained many important results in the theory of differential equations and analytic function theory.

[7] S. Banach (1892–1945) – Polish mathematician, one of the founders of functional analysis.

Proof. We shall take an arbitrary point $x_0 \in X$ and show that the sequence x_0, $x_1 = f(x_0)$, ..., $x_{n+1} = f(x_n)$,... is a Cauchy sequence. The mapping f is a contraction, so that by Eq. (9.20)

$$d(x_{n+1}, x_n) \le qd(x_n, x_{n-1}) \le \cdots \le q^n d(x_1, x_0)$$

and

$$\begin{aligned} d(x_{n+k}, x_n) &\le d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \le \\ &\le \left(q^n + q^{n+1} + \cdots + q^{n+k-1}\right) d(x_1, x_0) \le \frac{q^n}{1-q} d(x_1, x_0) . \end{aligned}$$

From this one can see that the sequence $x_0, x_1, \ldots, x_n, \ldots$ is indeed a Cauchy sequence.

The space (X, d) is complete, so that this sequence has a limit $\lim\limits_{n\to\infty} x_n = a \in X$.

It is clear from the definition of a contraction mapping that a contraction is always continuous, and therefore

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) = f(a) .$$

Thus a is a fixed point of the mapping f.

The mapping f cannot have a second fixed point, since the relations $a_i = f(a_i)$, $i = 1, 2$, imply, when we take account of (9.20), that

$$0 \le d(a_1, a_2) = d\big(f(a_1), f(a_2)\big) \le qd(a_1, a_2) ,$$

which is possible only if $d(a_1, a_2) = 0$, that is, $a_1 = a_2$.

Next, by passing to the limit as $k \to \infty$ in the relation

$$d(x_{n+k}, x_n) \le \frac{q^n}{1-q} d(x_1, x_0) ,$$

we find that

$$d(a, x_n) \le \frac{q^n}{1-q} d(x_1, x_0) . \square$$

The following proposition supplements this theorem.

Proposition. (Stability of the fixed point.) *Let (X, d) be a complete metric space and (Ω, τ) a topological space that will play the role of a parameter space in what follows.*

Suppose to each value of the parameter $t \in \Omega$ there corresponds a contraction mapping $f_t : X \to X$ of the space X into itself and that the following conditions hold.

a) *The family $\{f_t;\, t \in \Omega\}$ is uniformly contracting, that is, there exists q, $0 < q < 1$, such that each mapping f_t is a q-contraction.*

b) *For each $x \in X$ the mapping $f_t(x) : \Omega \to X$ is continuous as a function of t at some point $t_0 \in \Omega$, that is $\lim_{t\to t_0} f_t(x) = f_{t_0}(x)$.*

Then the solution $a(t) \in X$ of the equation $x = f_t(x)$ depends continuously on t at the point t_0, that is, $\lim_{t\to t_0} a(t) = a(t_0)$.

Proof. As was shown in the proof of the theorem, the solution $a(t)$ of the equation $x = f_t(x)$ can be obtained as the limit of the sequence $\{x_{n+1} = f_t(x_n);\ n = 0, 1, \dots\}$ starting from any point $x_0 \in X$. Let $x_0 = a(t_0) = f_{t_0}(a(t_0))$.

Taking account of the estimate (9.21) and condition a), we obtain

$$d(a(t), a(t_0)) = d(a(t), x_0) \le$$
$$\le \frac{1}{1-q} d(x_1, x_0) = \frac{1}{1-q} d\big(f_t(a(t_0)), f_{t_0}(a(t_0))\big) .$$

By condition b), the last term in this relation tends to zero as $t \to t_0$. Thus it has been proved that

$$\lim_{t\to t_0} d(a(t), a(t_0)) = 0 \text{ , that is, } \lim_{t\to t_0} a(t) = a(t_0) . \ \square$$

Example 1. As an important example of the application of the contraction mapping principle we shall prove, following Picard, an existence theorem for the solution of the differential equation $y'(x) = f(x, y(x))$ satisfying an initial condition $y(x_0) = y_0$.

If the function $f \in C(\mathbb{R}^2, \mathbb{R})$ is such that

$$|f(u, v_1) - f(u, v_2)| \le M|v_1 - v_2| ,$$

where M is a constant, then, for any initial condition

$$y(x_0) = y_0 , \tag{9.22}$$

there exists a neighborhood $U(x_0)$ of $x_0 \in \mathbb{R}$ and a unique function $y = y(x)$ defined in $U(x_0)$ satisfying the equation

$$y' = f(x, y) \tag{9.23}$$

and the initial condition (9.22).

Proof. Equation (9.23) and the condition (9.22) can be jointly written as a single relation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt . \tag{9.24}$$

Denoting the right-hand side of this equality by $A(y)$, we find that $A : C\big(V(x_0),\mathbb{R}\big) \to C\big(V(x_0),\mathbb{R}\big)$ is a mapping of the set of continuous functions defined on a neighborhood $V(x_0)$ of x_0 into itself. Regarding $C\big(V(x_0),\mathbb{R}\big)$ as a metric space with the uniform metric (see formula (9.6) from Sect. 9.1), we find that

$$\begin{aligned} d(Ay_1, Ay_2) &= \max_{x\in \bar V(x_0)} \left| \int_{x_0}^{x} f\big(t, y_1(t)\big)\, dt - \int_{x_0}^{x} f\big(t, y_2(t)\big)\, dt \right| \le \\ &\le \max_{x\in \bar V(x_0)} \left| \int_{x_0}^{x} M|y_1(t) - y_2(t)|\, dt \right| \le M|x - x_0|\, d(y_1, y_2)\,. \end{aligned}$$

If we assume that $|x - x_0| \le \frac{1}{2M}$, then the inequality

$$d(Ay_1, Ay_2) \le \frac{1}{2} d(y_1, y_2)$$

is fulfilled on the corresponding closed interval I, where $d(y_1, y_2) = \max\limits_{x\in I} |y_1(x) - y_2(x)|$. Thus we have a contraction mapping

$$A : C(I,\mathbb{R}) \to C(I,\mathbb{R})$$

of the complete metric space $\big(C(I,\mathbb{R}), d\big)$ (see Example 4 of Sect. 9.5) into itself, which by the contraction mapping principle must have a unique fixed point $y = Ay$. But this means that the function in $C(I,\mathbb{R})$ just found is the unique function defined on $I \ni x_0$ and satisfying Eq. (9.24). □

Example 2. As an illustration of what was just said, we shall seek a solution of the familiar equation

$$y' = y$$

with the initial condition (9.22) on the basis of the contraction mapping principle.

In this case

$$Ay = y_0 + \int_{x_0}^{x} y(t)\, dt\,,$$

and the principle is applicable at least for $|x - x_0| \le q < 1$.

Starting from the initial approximation $y(x) \equiv 0$, we construct successively the sequence $0, y_1 = A(0), \dots, y_{n+1}(t) = A\big(y_n(t)\big), \dots$ of approximations

$$
\begin{aligned}
y_1(t) &= y_0 \,, \\
y_2(t) &= y_0\bigl(1 + (x - x_0)\bigr) \,, \\
y_3(t) &= y_0\bigl(1 + (x - x_0) + \tfrac{1}{2}(x - x_0)^2\bigr) \,,
\end{aligned}
$$

....

$$y_{n+1}(t) = y_0\bigl(1 + (x - x_0) + \tfrac{1}{2!}(x - x_0)^2 + \cdots + \tfrac{1}{n!}(x - x_0)^n\bigr) \,,$$

....

from which it is already clear that

$$y(x) = y_0 e^{x - x_0} \,.$$

The fixed-point principle stated in the theorem above also goes by the name of the *contraction mapping principle.* It arose as a generalization of Picard's proof of the existence theorem for a solution of the differential equation (9.23), which was discussed in Example 1. The contraction mapping principle was stated in full generality by Banach.

Example 3. Newton's method of finding a root of the equation $f(x) = 0$. Suppose a real-valued function that is convex and has a positive derivative on a closed interval $[\alpha, \beta]$ assumes values of opposite signs at the endpoints of the interval. Then there is a unique point a in the interval at which $f(a) = 0$. In addition to the elementary method of finding the point a by successive bisection of the interval, there also exist more sophisticated and rapid methods of finding it, using the properties of the function f. Thus, in the present case, one may use the following method, proposed by Newton and called *Newton's method* or the *method of tangents.* Take an arbitrary point $x_0 \in [\alpha, \beta]$ and write the equation $y = f(x_0) + f'(x_0)(x - x_0)$ of the tangent to the graph of the function at the point $\bigl(x_0, f(x_0)\bigr)$. We then find the point $x_1 = x_0 - \bigl[f'(x_0)\bigr]^{-1} \cdot f(x_0)$ where the tangent intersects the x-axis (Fig. 9.3). We take x_1 as the first approximation of the root a and repeat this operation, replacing x_0 by x_1. In this way we obtain a sequence

$$x_{n+1} = x_n - \bigl[f'(x_n)\bigr]^{-1} \cdot f(x_n) \tag{9.25}$$

of points that, as one can verify, will tend monotonically to a in the present case.

In particular, if $f(x) = x^k - a$, that is, when we are seeking $\sqrt[k]{a}$, where $a > 0$, the recurrence relation (9.25) has the form

$$x_{n+1} = x_n - \frac{x_n^k - a}{k x_n^{k-1}} \,,$$

which for $k = 2$ becomes the familiar expression

$$x_{n+1} = \frac{1}{2}\Bigl(x_n + \frac{a}{x_n}\Bigr) \,.$$

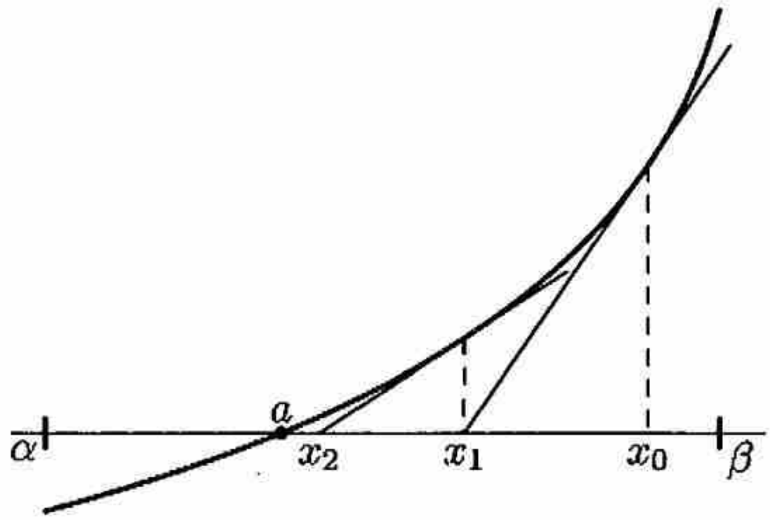

Fig. 9.3.

The method (9.25) for forming the sequence $\{x_n\}$ is called *Newton's method.*

If instead of the sequence (9.25) we consider the sequence obtained by the recurrence relation

$$x_{n+1} = x_n - \left[f'(x_0)\right]^{-1} \cdot f(x_n) , \tag{9.26}$$

we speak of the *modified Newton's method.*[8] The modification amounts to computing the derivative once and for all at the point x_0.

Consider the mapping

$$x \mapsto A(x) = x - \left[f'(x_0)\right]^{-1} \cdot f(x) . \tag{9.27}$$

By Lagrange's theorem

$$|A(x_2) - A(x_1)| = \left|\left[f'(x_0)\right]^{-1} \cdot f'(\xi)\right| \cdot |x_2 - x_1| ,$$

where ξ is a point lying between x_1 and x_2.

Thus, if the conditions

$$A(I) \subset I \tag{9.28}$$

and

$$\left|\left[f'(x_0)\right]^{-1} \cdot f'(x)\right| \leq q < 1 , \tag{9.29}$$

hold on some closed interval $I \subset \mathbb{R}$, then the mapping $A : I \to I$ defined by relation (9.27) is a contraction of this closed interval. Then by the general principle it has a unique fixed point on the interval. But, as can be seem from (9.27), the condition $A(a) = a$ is equivalent to $f(a) = 0$.

Hence, when conditions (9.28) and (9.29) hold for a function f, the modified Newton's method (9.26) leads to the required solution $x = a$ of the equation $f(x) = 0$ by the contraction mapping principle.

[8] In functional analysis it has numerous applications and is called the *Newton–Kantorovich method.* L.V. Kantorovich (1912–1986) – eminent Soviet mathematician, whose research in mathematical economics earned him the Nobel Prize.

9.7.1 Problems and Exercises

1. Show that condition (9.20) in the contraction mapping principle cannot be replaced by the weaker condition

$$d\Big(f(x_1), f(x_2)\Big) < d(x_1, x_2)\,.$$

2. a) Prove that if a mapping $f : X \to X$ of a complete metric space (X, d) into itself is such that some iteration of it $f^n : X \to X$ is a contraction, then f has a unique fixed point.

b) Verify that the mapping $A : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ in Example 2 is such that for any closed interval $I \subset \mathbb{R}$ some iteration A^n of the mapping A is a contraction.

c) Deduce from b) that the local solution $y = y_0 e^{x-x_0}$ found in Example 2 is actually a solution of the original equation on the entire real line.

3. a) Show that in the case of a function on $[\alpha, \beta]$ that is convex and has a positive derivative and assumes values of opposite signs at the endpoints, Newton's method really does give a sequence $\{x_n\}$ that converges to the point $a \in [\alpha, \beta]$ at which $f(a) = 0$.

b) Estimate the rate of convergence of the sequence (9.25) to the point a.

10 *Differential Calculus from a more General Point of View

10.1 Normed Vector Spaces

Differentiation is the process of finding the best local linear approximation of a function. For that reason any reasonably general theory of differentiation must be based on elementary ideas connected with linear functions. From the course in algebra the reader is well acquainted with the concept of a *vector space*, as well as linear dependence and independence of systems of vectors, bases and dimension of a vector space, vector subspaces, and so forth. In the present section we shall present vector spaces with a norm, or as they are described, *normed vector spaces*, which are widely used in analysis. We begin, however, with some examples of vector spaces.

10.1.1 Some Examples of Vector Spaces in Analysis

Example 1. The real vector space $\mathbb{R}^n$ and the complex vector space $\mathbb{C}^n$ are classical examples of vector spaces of dimension n over the fields of real and complex numbers respectively.

Example 2. In analysis, besides the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ exhibited in Example 1, we encounter the space closest to them, which is the space ℓ of sequences $x = (x^1, \dots, x^n, \dots)$ of real or complex numbers. The vector-space operations in ℓ, as in $\mathbb{R}^n$ and $\mathbb{C}^n$, are carried out coordinatewise. One peculiarity of this space, when compared with $\mathbb{R}^n$ or $\mathbb{C}^n$ is that any finite subsystem of the countable system of vectors $\{x_i = (0, \dots, 0, x^i = 1, 0, \dots), i \in \mathbb{N}\}$ is linearly independent, that is, ℓ is an infinite-dimensional vector space (of countable dimension in the present case).

The set of finite sequences (all of whose terms are zero from some point on) is a vector subspace $\underset{0}{\ell}$ of the space ℓ, also infinite-dimensional.

Example 3. Let $F[a, b]$ be the set of numerical-valued (real- or complex-valued) functions defined on the closed interval $[a, b]$. This set is a vector space over the corresponding number field with respect to the operations of addition of functions and multiplication of a function by a number.

The set of functions of the form

$$e_\tau(x) = \begin{cases} 0\,, & \text{if} \quad x \in [a,b] \quad \text{and} \quad x \neq \tau\,, \\ 1\,, & \text{if} \quad x \in [a,b] \quad \text{and} \quad x = \tau \end{cases}$$

is a continuously indexed system of linearly independent vectors in $F[a,b]$.

The set $C[a,b]$ of continuous functions is obviously a subspace of the space $F[a,b]$ just constructed.

Example 4. If X_1 and X_2 are two vector spaces over the same field, there is a natural way of introducing a vector-space structure into their direct product $X_1 \times X_2$, namely by carrying out the vector-space operations on elements $x = (x_1, x_2) \in X_1 \times X_2$ coordinatewise.

Similarly one can introduce a vector-space structure into the direct product $X_1 \times \cdots \times X_n$ of any finite set of vector spaces. This is completely analogous to the cases of $\mathbb{R}^n$ and $\mathbb{C}^n$.

10.1.2 Norms in Vector Spaces

We begin with the basic definition.

Definition 1. Let X be a vector space over the field of real or complex numbers.

A function $\|\ \| : X \to \mathbb{R}$ assigning to each vector $x \in X$ a real number $\|x\|$ is called a *norm* in the vector space X if it satisfies the following three conditions:

a) $\|x\| = 0 \Leftrightarrow x = 0$ (nondegeneracy);
b) $\|\lambda x\| = |\lambda|\,\|x\|$ (homogeneity);
c) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ (the triangle inequality).

Definition 2. A vector space with a norm defined on it is called a *normed vector space.*

Definition 3. The value of the norm at a vector is called the *norm of that vector.*

The norm of a vector is always nonnegative and, as can be seen by a), equals zero only for the zero vector.

Proof. Indeed, by c), taking account of a) and b), we obtain for every $x \in X$,

$$0 = \|0\| = \|x + (-x)\| \leq \|x\| + \|-x\| = \|x\| + |-1|\,\|x\| = 2\|x\|\,. \ \square$$

By induction, condition c) implies the following general inequality.

$$\|x_1 + \cdots + x_n\| \leq \|x_1\| + \cdots + \|x_n\|\,, \tag{10.1}$$

and taking account of b), one can easily deduce from c) the following useful inequality.

$$\big|\|x_1\| - \|x_2\|\big| \le \|x_1 - x_2\| \,. \tag{10.2}$$

Every normed vector space has a natural metric

$$d(x_1, x_2) = \|x_1 - x_2\| \,. \tag{10.3}$$

The fact that the function $d(x_1, x_2)$ just defined satisfies the axioms for a metric follows immediately from the properties of the norm. Because of the vector-space structure in X the metric d in X has two additional special properties:

$$d(x_1 + x, x_2 + x) = \|(x_1 + x) - (x_2 + x)\| = \|x_1 - x_2\| = d(x_1, x_2) \,,$$

that is, the metric is translation-invariant, and

$$d(\lambda x_1, \lambda x_2) = \|\lambda x_1 - \lambda x_2\| = \|\lambda(x_1 - x_2)\| = |\lambda|\,\|x_1 - x_2\| = |\lambda|\, d(x_1, x_2) \,,$$

that is, it is homogeneous.

Definition 4. If a normed vector space is complete as a metric space with the natural metric (10.3), it is called a *complete normed vector space* or *Banach space*.

Example 5. If for $p \ge 1$ we set

$$\|x\|_p := \left(\sum_{i=1}^{n} |x^i|^p\right)^{\frac{1}{p}} \tag{10.4}$$

for $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, it follows from Minkowski's inequality that we obtain a norm on $\mathbb{R}^n$. The space $\mathbb{R}^n$ endowed with this norm will be denoted $\mathbb{R}^n_p$.

One can verify that

$$\|x\|_{p_2} \le \|x\|_{p_1} \,, \text{ if } 1 \le p_1 \le p_2 \,, \tag{10.5}$$

and that

$$\|x\|_p \to \max\left\{|x^1|, \dots, |x^n|\right\} \tag{10.6}$$

as $p \to +\infty$. Thus, it is natural to set

$$\|x\|_\infty := \max\left\{|x^1|, \dots, |x^n|\right\} \,. \tag{10.7}$$

It then follows from (10.4) and (10.5) that

$$\|x\|_\infty \le \|x\|_p \le \|x\|_1 \le n\|x\|_\infty \text{ for } p \ge 1 \,. \tag{10.8}$$

It is clear from this inequality, as in fact it is from the very definition of the norm $\|x\|_p$ in Eq. (10.4), that $\mathbb{R}^n_p$ is a complete normed vector space.

Example 6. The preceding example can be usefully generalized as follows. If $X = X_1 \times \cdots \times X_n$ is the direct product of normed vector spaces, one can introduce the norm of a vector $x = (x_1, \ldots, x_n)$ in the direct product by setting

$$\|x\|_p := \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}, \quad p \geq 1, \tag{10.9}$$

where $\|x_i\|$ is the norm of the vector $x_i \in X_i$.

Naturally, inequalities (10.8) remain valid in this case as well.

From now on, when the direct product of normed spaces is considered, unless the contrary is explicitly stated, it is assumed that the norm is defined in accordance with formula (10.9) (including the case $p = +\infty$).

Example 7. Let $p \geq 1$. We denote by ℓ_p the set of sequences $x = (x^1, \ldots, x^n, \ldots)$ of real or complex numbers such that the series $\sum_{n=1}^{\infty} |x^n|^p$ converges, and for $x \in \ell_p$ we set

$$\|x\|_p := \left(\sum_{n=1}^{\infty} |x^n|^p \right)^{\frac{1}{p}}. \tag{10.10}$$

Using Minkowski's inequality, one can easily see that ℓ_p is a normed vector space with respect to the standard vector-space operations and the norm (10.10). This is an infinite-dimensional space with respect to which $\mathbb{R}^n_p$ is a vector subspace of finite dimension.

All the inequalities (10.8) except the last are valid for the norm (10.10). It is not difficult to verify that ℓ_p is a Banach space.

Example 8. In the vector space $C[a, b]$ of numerical-valued functions that are continuous on the closed interval $[a, b]$, one usually considers the following norm:

$$\|f\| := \max_{x \in [a,b]} |f(x)|. \tag{10.11}$$

We leave the verification of the norm axioms to the reader. We remark that this norm generates a metric on $C[a, b]$ that is already familiar to us (see Sect. 9.5), and we know that the metric space that thereby arises is complete. Thus the vector space $C[a, b]$ with the norm (10.11) is a Banach space.

Example 9. One can also introduce another norm in $C[a, b]$

$$\|f\|_p := \left(\int_a^b |f|^p(x)\, dx \right)^{\frac{1}{p}}, \quad p \geq 1, \tag{10.12}$$

which becomes (10.11) as $p \to +\infty$.

It is easy to see (for example, Sect. 9.5) that the space $C[a, b]$ with the norm (10.12) is not complete for $1 \leq p < +\infty$.

10.1.3 Inner Products in Vector Spaces

An important class of normed spaces is formed by the spaces with an inner product. They are a direct generalization of Euclidean spaces.

We recall their definition.

Definition 5. We say that a *Hermitian form* is defined in a vector space X (over the field of complex numbers) if there exists a mapping $\langle\, ,\rangle : X\times X \to \mathbb{C}$ having the following properties:

a) $\langle x_1, x_2\rangle = \overline{\langle x_2, x_1\rangle}$,

b) $\langle \lambda x_1, x_2\rangle = \lambda\langle x_1, x_2\rangle$,

c) $\langle x_1 + x_2, x_3\rangle = \langle x_1, x_3\rangle + \langle x_2, x_3\rangle$,

where x_1, x_2, x_3 are vectors in X and $\lambda \in \mathbb{C}$.

It follows from a), b), and c), for example, that

$$\langle x_1, \lambda x_2\rangle = \overline{\langle \lambda x_2, x_1\rangle} = \overline{\lambda\langle x_2, x_1\rangle} = \overline{\lambda}\,\overline{\langle x_2, x_1\rangle} = \overline{\lambda}\langle x_1, x_2\rangle\ ;$$
$$\langle x_1, x_2 + x_3\rangle = \overline{\langle x_2 + x_3, x_1\rangle} = \overline{\langle x_2, x_1\rangle} + \overline{\langle x_3, x_1\rangle} = \langle x_1, x_2\rangle + \langle x_1, x_3\rangle\ ;$$
$$\langle x, x\rangle = \overline{\langle x, x\rangle}\ , \text{ that is, } \langle x, x\rangle \text{ is a real number.}$$

A Hermitian form is called *nonnegative* if

d) $\langle x, x\rangle \geq 0$

and *nondegenerate* if

e) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

If X is a vector space over the field of real numbers, one must of course consider a real-valued form $\langle x_1, x_2\rangle$. In this case a) can be replaced by $\langle x_1, x_2\rangle = \langle x_2, x_1\rangle$, which means that the form is symmetric with respect to its vector arguments x_1 and x_2.

An example of such a form is the dot product familiar from analytic geometry for vectors in three-dimensional Euclidean space. In connection with this analogy we make the following definition.

Definition 6. A nondegenerate nonnegative Hermitian form in a vector space is called an *inner product* in the space.

Example 10. An inner product of vectors $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$ in $\mathbb{R}^n$ can be defined by setting

$$\langle x, y\rangle := \sum_{i=1}^{n} x^i y^i\ , \tag{10.13}$$

and in $\mathbb{C}^n$ by setting

$$\langle x, y\rangle := \sum_{i=1}^{n} x^i \overline{y^i}\ . \tag{10.14}$$

Example 11. In ℓ_2 the inner product of the vectors x and y can be defined as

$$\langle x, y\rangle := \sum_{i=1}^{\infty} x^i \overline{y^i} ,$$

The series in this expression converges absolutely since

$$2\sum_{i=1}^{\infty} |x^i \overline{y^i}| \le \sum_{i=1}^{\infty} |x^i|^2 + \sum_{i=1}^{\infty} |y^i|^2 .$$

Example 12. An inner product can be defined in $C[a, b]$ by the formula

$$\langle f, g\rangle := \int_a^b (f \cdot \bar{g})(x)\, dx . \tag{10.15}$$

It follows easily from properties of the integral that all the requirements for an inner product are satisfied in this case.

The following important inequality, known as the *Cauchy–Bunyakovskii inequality*, holds for the inner product:

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle \cdot \langle y, y\rangle , \tag{10.16}$$

where equality holds if and only if the vectors x and y are collinear.

Proof. Indeed, let $a = \langle x, x\rangle$, $b = \langle x, y\rangle$, and $c = \langle y, y\rangle$. By hypothesis $a \ge 0$ and $c \ge 0$. If $c > 0$, the inequalities

$$0 \le \langle x + \lambda y, x + \lambda y\rangle = a + \bar{b}\lambda + b\bar{\lambda} + c\lambda\bar{\lambda}$$

with $\lambda = -\frac{b}{c}$ imply

$$0 \le a - \frac{\bar{b}b}{c} - \frac{b\bar{b}}{c} + \frac{b\bar{b}}{c}$$

or

$$0 \le ac - b\bar{b} = ac - |b|^2 , \tag{10.17}$$

which is the same as (10.16).

The case $a > 0$ can be handled similarly.

If $a = c = 0$, then, setting $\lambda = -b$ in (10.17), we find $0 \le -\bar{b}b - b\bar{b} = -2|b|^2$, that is, $b = 0$, and (10.16) is again true.

If x and y are not collinear, then $0 < \langle x + \lambda y, x + \lambda y\rangle$ and consequently inequality (10.16) is a strict inequality in this case. But if x and y are collinear, it becomes equality as one can easily verify. □

A vector space with an inner product has a natural norm:

$$\|x\| := \sqrt{\langle x, x\rangle} \tag{10.18}$$

and metric

$$d(x,y) := \|x-y\| .$$

Using the Cauchy–Bunyakovskii inequality, we verify that if $\langle x, y\rangle$ is a nondegenerate nonnegative Hermitian form, then formula (10.18) does indeed define a norm.

Proof. In fact,

$$\|x\| = \sqrt{\langle x, x\rangle} = 0 \Leftrightarrow x = 0 ,$$

since the form $\langle x, y\rangle$ is nondegenerate.

Next,

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{\lambda\bar{\lambda}\langle x, x\rangle} = |\lambda|\sqrt{\langle x, x\rangle} = |\lambda|\,\|x\| .$$

We verify finally that the triangle inequality holds:

$$\|x+y\| \leq \|x\| + \|y\| .$$

Thus, we need to show that

$$\sqrt{\langle x+y, x+y\rangle} \leq \sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle} ,$$

or, after we square and cancel, that

$$\langle x, y\rangle + \langle y, x\rangle \leq 2\sqrt{\langle x, x\rangle \cdot \langle y, y\rangle} .$$

But

$$\langle x, y\rangle + \langle y, x\rangle = \langle x, y\rangle + \overline{\langle x, y\rangle} = 2\mathrm{Re}\,\langle x, y\rangle \leq 2|\langle x, y\rangle| ,$$

and the inequality to be proved now follows immediately from the Cauchy–Bunyakovskii inequality (10.16). □

In conclusion we note that finite-dimensional vector spaces with an inner product are usually called *Euclidean* or *Hermitian* (*unitary*) spaces according as the field of scalars is $\mathbb{R}$ or $\mathbb{C}$ respectively. If a normed vector space is infinite-dimensional, it is called a *Hilbert space* if it is complete in the metric induced by the natural norm and a *pre-Hilbert space* otherwise.

10.1.4 Problems and Exercises

1. a) Show that if a translation-invariant homogeneous metric $d(x_1, x_2)$ is defined in a vector space X, then X can be normed by setting $\|x\| = d(0, x)$.

b) Verify that the norm in a vector space X is a continuous function with respect to the topology induced by the natural metric (10.3).

c) Prove that if X is a finite-dimensional vector space and $\|x\|$ and $\|x\|'$ are two norms on X, then one can find positive numbers M, N such that

$$M\|x\| \le \|x\|' \le N\|x\| \qquad (10.19)$$

for any vector $x \in X$.

d) Using the example of the norms $\|x\|_1$ and $\|x\|_\infty$ in the space ℓ, verify that the preceding inequality generally does not hold in infinite-dimensional spaces.

2. a) Prove inequality (10.5).

b) Verify relation (10.6).

c) Show that as $p \to +\infty$ the quantity $\|f\|_p$ defined by formula (10.12) tends to the quantity $\|f\|$ given by formula (10.11).

3. a) Verify that the normed space ℓ_p considered in Example 7 is complete.

b) Show that the subspace of ℓ_p consisting of finite sequences (ending in zeros) is not a Banach space.

4. a) Verify that relations (10.11) and (10.12) define a norm in the space $C[a, b]$ and convince yourself that a complete normed space is obtained in one of these cases but not in the other.

b) Does formula (10.12) define a norm in the space $\mathcal{R}[a, b]$ of Riemann-integrable functions?

c) What factorization (identification) must one make in $\mathcal{R}[a, b]$ so that the quantity defined by (10.12) will be a norm in the resulting vector space?

5. a) Verify that formulas (10.13)–(10.15) do indeed define an inner product in the corresponding vector spaces.

b) Is the form defined by formula (10.15) an inner product in the space $\mathcal{R}[a, b]$ of Riemann-integrable functions?

c) Which functions in $\mathcal{R}[a, b]$ must be identified so that the answer to part b) will be positive in the quotient space of equivalence classes?

6. Using the Cauchy–Bunyakovskii inequality, find the greatest lower bound of the values of the product $\left(\int_a^b f(x)\,dx\right)\left(\int_a^b (1/f)(x)\,dx\right)$ on the set of continuous real-valued functions that do not vanish on the closed interval $[a, b]$.

10.2 Linear and Multilinear Transformations

10.2.1 Definitions and Examples

We begin by recalling the basic definition.

Definition 1. If X and Y are vector spaces over the same field (in our case, either $\mathbb{R}$ or $\mathbb{C}$), a mapping $A : X \to Y$ is *linear* if the equalities

$$A(x_1 + x_2) = A(x_1) + A(x_2)\ ,$$
$$A(\lambda x) = \lambda A(x)$$

hold for any vectors x, x_1, x_2 in X and any number λ in the field of scalars.

For a linear transformation $A : X \to Y$ we often write Ax instead of $A(x)$.

Definition 2. A mapping $A : X_1 \times \cdots \times X_n \to Y$ of the direct product of the vector spaces $X_1, \dots, X_n$ into the vector space Y is *multilinear* (*n-linear*) if the mapping $y = A(x_1, \dots, x_n)$ is linear with respect to each variable for all fixed values of the other variables.

The set of n-linear mappings $A : X_1 \times \cdots \times X_n \to Y$ will be denoted $\mathcal{L}(X_1, \dots, X_n; Y)$.

In particular for $n = 1$ we obtain the set $\mathcal{L}(X; Y)$ of linear mappings from $X_1 = X$ into Y.

For $n = 2$ a multilinear mapping is called *bilinear*, for $n = 3$, *trilinear*, and so forth.

One should not confuse an n-linear mapping $A \in \mathcal{L}(X_1, \dots, X_n; Y)$ with a linear mapping $A \in \mathcal{L}(X; Y)$ of the vector space $X = X_1 \times \cdots \times X_n$ (in this connection see Examples 9–11 below).

If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, linear and multilinear mappings are usually called linear or multilinear *functionals*. When Y is an arbitrary vector space, a linear mapping $A : X \to Y$ is usually called a *linear transformation* from X into Y, and a *linear operator* in the special case when $X = Y$.

Let us consider some examples of linear mappings.

Example 1. Let $\underset{0}{\ell}$ be the vector space of finite numerical sequences. We define a transformation $A : \underset{0}{\ell} \to \underset{0}{\ell}$ as follows:

$$A\big((x_1, x_2, \dots, x_n, 0, \dots)\big) := (1x_1, 2x_2, \dots, nx_n, 0, \dots)\ .$$

Example 2. We define the functional $A : C[a, b] \to \mathbb{R}$ by the relation

$$A(f) := f(x_0)\ ,$$

where $f \in C\big([a, b], \mathbb{R}\big)$ and x_0 is a fixed point of the closed interval $[a, b]$.

Example 3. We define the functional $A : C([a,b],\mathbb{R}) \to \mathbb{R}$ by the relation

$$A(f) := \int_a^b f(x)\,dx\,.$$

Example 4. We define the transformation $A : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by the formula

$$A(f) := \int_a^x f(t)\,dt\,,$$

where x is a point ranging over the closed interval $[a,b]$.

All of these transformations are obviously linear.

Let us now consider some familiar examples of multilinear mappings.

Example 5. The usual product $(x_1,\dots,x_n) \mapsto x_1 \cdot \ldots \cdot x_n$ of n real numbers is a typical example of an n-linear functional $A \in \mathcal{L}(\underbrace{\mathbb{R},\dots,\mathbb{R}}_{n};\mathbb{R})$.

Example 6. The inner product $(x_1,x_2) \overset{A}{\longmapsto} \langle x_1,x_2\rangle$ in a Euclidean vector space over the field $\mathbb{R}$ is a bilinear function.

Example 7. The cross product $(x_1,x_2) \overset{A}{\longmapsto} [x_1,x_2]$ of vectors in three-dimensional Euclidean space E^3 is a bilinear transformation, that is, $A \in \mathcal{L}(E^3,E^3;E^3)$.

Example 8. If X is a finite-dimensional vector space over the field $\mathbb{R}$, $\{e_1,\dots,e_n\}$ is a basis in X, and $x = x^i e_i$ is the coordinate representation of the vector $x \in X$, then, setting

$$A(x_1,\dots,x_n) = \det\begin{pmatrix} x_1^1 & \cdots & x_1^n \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ x_n^1 & \cdots & x_n^n \end{pmatrix},$$

we obtain an n-linear function $A : X^n \to \mathbb{R}$.

As a useful supplement to the examples just given, we investigate in addition the structure of the linear mappings of a product of vector spaces into a product of vector spaces.

Example 9. Let $X = X_1 \times \cdots \times X_m$ be the vector space that is the direct product of the spaces $X_1,\dots,X_m$, and let $A : X \to Y$ be a linear mapping of X into a vector space Y. Representing every vector $x = (x_1,\dots,x_m) \in X$ in the form

$$x = (x_1, \dots, x_m) =$$
$$= (x_1, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \dots + (0, \dots, 0, x_m) \quad (10.20)$$

and setting

$$A_i(x_i) := A\big((0, \dots, 0, x_i, 0, \dots, 0)\big) \quad (10.21)$$

for $x_i \in X_i$, $i = \{1, \dots, m\}$, we observe that the mappings $A_i : X_i \to Y$ are linear and that

$$A(x) = A_1(x_1) + \dots + A_m(x_m)\,. \quad (10.22)$$

Since the mapping $A : X = X_1 \times \dots \times X_m \to Y$ is obviously linear for any linear mappings $A_i : X_i \to Y$, we have shown that formula (10.22) gives the general form of any linear mapping $A \in \mathcal{L}(X = X_1 \times \dots \times X_m; Y)$.

Example 10. Starting from the definition of the direct product $Y = Y_1 \times \dots \times Y_n$ of the vector spaces $Y_1, \dots, Y_n$ and the definition of a linear mapping $A : X \to Y$, one can easily see that any linear mapping

$$A : X \to Y = Y_1 \times \dots \times Y_n$$

has the form $x \mapsto Ax = (A_1 x, \dots, A_n x) = (y_1, \dots, y_n) = y \in Y$, where $A_i : X \to Y_i$ are linear mappings.

Example 11. Combining Examples 9 and 10, we conclude that any linear mapping

$$A : X_1 \times \dots \times X_m = X \to Y = Y_1 \times \dots \times Y_n$$

of the direct product $X = X_1 \times \dots \times X_m$ of vector spaces into another direct product $Y = Y_1 \times \dots \times Y_n$ has the form

$$y = \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \dots & \dots & \dots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_m \end{pmatrix} = Ax\,, \quad (10.23)$$

where $A_{ij} : X_j \to Y_i$ are linear mappings.

In particular, if $X_1 = X_2 = \dots = X_m = \mathbb{R}$ and $Y_1 = Y_2 = \dots = Y_n = \mathbb{R}$, then $A_{ij} : X_j \to Y_i$ are the linear mappings $\mathbb{R} \ni x \mapsto a_{ij}x \in \mathbb{R}$, each of which is given by a single number a_{ij}. Thus in this case relation (10.23) becomes the familiar numerical notation for a linear mapping $A : \mathbb{R}^m \to \mathbb{R}^n$.

10.2.2 The Norm of a Transformation

Definition 3. Let $A : X_1 \times \dots \times X_n \to Y$ be a multilinear transformation mapping the direct product of the normed vector spaces $X_1, \dots, X_n$ into a normed space Y.

The quantity

$$\|A\| := \sup_{\substack{x_1,\dots,x_n \\ x_i \neq 0}} \frac{|A(x_1,\dots,x_n)|_Y}{|x_1|_{X_1} \times \cdots \times |x_n|_{X_n}}, \tag{10.24}$$

where the supremum is taken over all sets $x_1,\dots,x_n$ of nonzero vectors in the spaces $X_1,\dots,X_n$, is called the *norm* of the multilinear transformation A.

On the right-hand side of Eq. (10.24) we have denoted the norm of a vector x by the symbol $|\cdot|$ subscripted by the symbol for the normed vector space to which the vector belongs, rather than the usual symbol $\|\cdot\|$ for the norm of a vector. From now on we shall adhere to this notation for the norm of a vector; and, where no confusion can arise, we shall omit the symbol for the vector space, taking for granted that the norm (absolute value) of a vector is always computed in the space to which it belongs. In this way we hope to introduce for the time being some distinction in the notation for the norm of a vector and the norm of a linear or multilinear transformation acting on a normed vector space.

Using the properties of the norm of a vector and the properties of a multilinear transformation, one can rewrite formula (10.24) as follows:

$$\|A\| = \sup_{\substack{x_1,\dots,x_n \\ x_i \neq 0}} \left| A\Big(\frac{x_1}{|x_1|},\dots,\frac{x_n}{|x_n|}\Big)\right| = \sup_{e_1,\dots,e_n} |A(e_1,\dots,e_n)|, \tag{10.25}$$

where the last supremum extends over all sets $e_1,\dots,e_n$ of unit vectors in the spaces $X_1,\dots,X_n$ respectively (that is, $|e_i| = 1$, $i = 1,\dots,n$).

In particular, for a linear transformation $A : X \to Y$, from (10.24) and (10.25) we obtain

$$\|A\| = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \sup_{|e|=1} |Ae|. \tag{10.26}$$

It follows from Definition 3 for the norm of a multilinear transformation A that if $\|A\| < \infty$, then the inequality

$$|A(x_1,\dots,x_n)| \leq \|A\|\,|x_1| \times \cdots \times |x_n| \tag{10.27}$$

holds for any vectors $x_i \in X_i$, $i = 1,\dots,n$.

In particular, for a linear transformation we obtain

$$|Ax| \leq \|A\|\,|x|. \tag{10.28}$$

In addition, it follows from Definition 3 that if the norm of a multilinear transformation is finite, it is the greatest lower bound of all numbers M for which the inequality

$$|A(x_1,\dots,x_n)| \leq M|x_1| \times \cdots \times |x_n| \tag{10.29}$$

holds for all values of $x_i \in X_i$, $i = 1,\dots,n$.

Definition 4. A multilinear transformation $A : X_1 \times \cdots \times X_n \to Y$ is *bounded* if there exists $M \in \mathbb{R}$ such that inequality (10.29) holds for all values of $x_1, \ldots, x_n$ in the spaces $X_1, \ldots, X_n$ respectively.

Thus the bounded transformations are precisely those that have a finite norm.

On the basis of relation (10.26) one can easily understand the geometric meaning of the norm of a linear transformation in the familiar case $A : \mathbb{R}^m \to \mathbb{R}^n$. In this case the unit sphere in $\mathbb{R}^m$ maps under the transformation A into some ellipsoid in $\mathbb{R}^n$ whose center is at the origin. Hence the norm of A in this case is simply the largest of the semiaxes of the ellipsoid.

On the other hand, one can also interpret the norm of a linear transformation as the least upper bound of the coefficients of dilation of vectors under the mapping, as can be seen from the first equality in (10.26).

It is not difficult to prove that for mappings of finite-dimensional spaces the norm of a multilinear transformation is always finite, and hence in particular the norm of a linear transformation is always finite. This is no longer true in the case of infinite-dimensional spaces, as can be seen from the first of the following examples.

Let us compute the norms of the transformations considered in Examples 1–8.

Example 1$'$. If we regard $\underset{0}{\ell}$ as a subspace of the normed space ℓ_p, in which the vector $e_n = (\underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots)$ has unit norm, then, since $Ae_n = ne_n$, it s clear that $\|A\| = \infty$.

Example 2$'$. If $|f| = \max\limits_{a \le x \le b} |f(x)| \le 1$, then $|Af| = |f(x_0)| \le 1$, and $|Af| = 1$ f $f(x_0) = 1$, so that $\|A\| = 1$.

We remark that if we introduce, for example, the integral norm

$$|f| = \int\limits_a^b |f|(x)\, dx$$

n the same vector space $C\big([a,b], \mathbb{R}\big)$, the result of computing $\|A\|$ may change onsiderably. Indeed, set $[a,b] = [0,1]$ and $x_0 = 1$. The integral norm of the unction $f_n = x^n$ on $[0,1]$ is obviously $\frac{1}{n+1}$, while $Af_n = Ax^n = x^n\big|_{x=1} = 1$. t follows that $\|A\| = \infty$ in this case.

Throughout what follows, unless the contrary is explicitly stated, the pace $C\big([a,b], \mathbb{R}\big)$ is assumed to have the norm defined by the maximum of he absolute value of the function on the closed interval $[a,b]$.

Example 3′. If $|f| = \max\limits_{a \le x \le b} |f(x)| \le 1$, then

$$|Af| = \left| \int\limits_a^b f(x)\, dx \right| \le \int\limits_a^b |f|(x)\, dx \le \int\limits_a^b 1\, dx = b - a \,.$$

But for $f(x) \equiv 1$, we obtain $|A1| = b - a$, and therefore $\|A\| = b - a$.

Example 4′. If $|f| = \max\limits_{a \le x \le b} |f(x)| \le 1$, then

$$\max_{a \le x \le b} \left| \int\limits_a^x f(t)\, dt \right| \le \max_{a \le x \le b} \int_a^x |f|(t)\, dt \le \max_{a \le x \le b} (x - a) = b - a \,.$$

But for $|f(t) \equiv 1$, we obtain

$$\max_{a \le x \le b} \int\limits_a^x 1\, dt = b - a \,,$$

and therefore in this example $\|A\| = b - a$.

Example 5′. We obtain immediately from Definition 3 that $\|A\| = 1$ in this case.

Example 6′. By the Cauchy–Bunyakovskii inequality

$$|\langle x_1, x_2 \rangle| \le |x_1| \cdot |x_2| \,,$$

and if $x_1 = x_2$, this inequality becomes equality. Hence $\|A\| = 1$.

Example 7′. We know that

$$\big|[x_1, x_2]\big| = |x_1|\, |x_2| \sin\varphi \,,$$

where φ is the angle between the vectors x_1 and x_2, and therefore $\|A\| \le 1$. At the same time, if the vectors x_1 and x_2 are orthogonal, then $\sin\varphi = 1$. Thus $\|A\| = 1$.

Example 8′. If we assume that the vectors lie in a Euclidean space of dimension n, we note that $A(x_1, \dots, x_n) = \det(x_1, \dots, x_n)$ is the volume of the parallelepiped spanned by the vectors $x_1, \dots, x_n$, and this volume is maximal if the vectors $x_1, \dots, x_n$ are made pairwise orthogonal while keeping their lengths constant.

Thus,

$$|\det(x_1, \dots, x_n)| \le |x_1| \cdot \ldots \cdot |x_n| \,,$$

equality holding for orthogonal vectors. Hence in this case $\|A\| = 1$.

Let us now estimate the norms of the operators studied in Examples 9–11. We shall assume that in the direct product $X = X_1 \times \cdots \times X_m$ of the normed spaces $X_1, \ldots, X_m$ the norm of the vector $x = (x_1, \ldots, x_m)$ is introduced in accordance with the convention in Sect. 10.1 (Example 6).

Example 9$'$. Defining a linear transformation

$$A : X_1 \times \cdots \times X_m = X \to Y ,$$

as has been shown, is equivalent to defining the m linear transformations $A_i : X_i \to Y$ given by the relations $A_i x_i = A\big((0, \ldots, 0, x_i, 0, \ldots, 0)\big)$, $i = 1, \ldots, m$. When this is done, formula (10.22) holds, by virtue of which

$$|Ax|_Y \le \sum_{i=1}^{m} |A_i x_i|_Y \le \sum_{i=1}^{m} \|A_i\| \, |x_i|_{X_i} \le \Big(\sum_{i=1}^{m} \|A_i\| \Big) |x|_X .$$

Thus we have shown that

$$\|A\| \le \sum_{i=1}^{m} \|A_i\| .$$

On the other hand, since

$$|A_i x_i| = \big|A\big((0, \ldots, 0, x_i, 0, \ldots, 0)\big)\big| \le$$
$$\le \|A\| \, \big|(0, \ldots, 0, x_i, 0, \ldots, 0)\big|_X = \|A\| \, |x_i|_{X_i} ,$$

we can conclude that the estimate

$$\|A_i\| \le \|A\|$$

also holds for all $i = 1, \ldots, m$.

Example 10$'$. Taking account of the norm introduce in $Y = Y_1 \times \cdots \times Y_n$, in this case we immediately obtain the two-sided estimates

$$\|A_i\| \le \|A\| \le \sum_{i=1}^{n} \|A_i\| .$$

Example 11$'$. Taking account of the results of Examples 9 and 10, one can conclude that

$$\|A_{ij}\| \le \|A\| \le \sum_{i=1}^{m} \sum_{j=1}^{n} \|A_{ij}\| .$$

10.2.3 The Space of Continuous Transformations

From now on we shall not be interested in all linear or multilinear transformations, only continuous ones. In this connection it is useful to keep in mind the following proposition.

Proposition 1. *For a multilinear transformation $A : X_1 \times \cdots \times X_n \to Y$ mapping a product of normed spaces $X_1, \dots, X_n$ into a normed space Y the following conditions are equivalent:*

a) A has a finite norm,
b) A is a bounded transformation,
c) A is a continuous transformation,
d) A is continuous at the point $(0, \dots, 0) \in X_1 \times \cdots \times X_n$.

Proof. We prove a closed chain of implications $a) \Rightarrow b) \Rightarrow c) \Rightarrow d) \Rightarrow a)$.

It is obvious from relation (10.27) that $a) \Rightarrow b)$.

Let us verify that $b) \Rightarrow c)$, that is, that (10.29) implies that the operator A is continuous. Indeed, taking account of the multilinearity of A, we can write that

$$\begin{aligned}
&A(x_1 + h_1, x_2 + h_2, \dots, x_n + h_n) - A(x_1, x_2, \dots, x_n) = \\
&\qquad = A(h_1, x_2, \dots, x_n) + \cdots + A(x_1, x_2, \dots, x_{n-1}, h_n) = \\
&\qquad + A(h_1, h_2, x_3, \dots, x_n) + \cdots + A(x_1, \dots, x_{n-2}, h_{n-1}, h_n) + \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad\qquad + A(h_1, \dots, h_n) .
\end{aligned}$$

From (10.29) we now obtain the estimate

$$\begin{aligned}
&|A(x_1 + h_1, x_2 + h_2, \dots, x_n + h_n) - A(x_1, x_2, \dots, x_n) \le \\
&\qquad \le M\big(|h_1| \cdot |x_2| \cdot \ldots \cdot |x_n| + \cdots + |x_1| \cdot |x_2| \cdot \ldots \cdot |x_{n-1}| \cdot |h_n| + \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad\qquad + |h_1| \cdot \ldots \cdot |h_n|\big) ,
\end{aligned}$$

from which it follows that A is continuous at each point $(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$.

In particular, if $(x_1, \dots, x_n) = (0, \dots, 0)$ we obtain d) from c).

It remains to be shown that $d) \Rightarrow a)$.

Given $\varepsilon > 0$ we find $\delta = \delta(\varepsilon) > 0$ such that $|A(x_1, \dots, x_n)| < \varepsilon$ when $\max\{|x_1|, \dots, |x_n|\} < \delta$. Then for any set $e_1, \dots, e_n$ of unit vectors we obtain

$$|A(e_1, \dots, e_n)| = \frac{1}{\delta^n} |A(\delta e_1, \dots, \delta e_n)| < \frac{\varepsilon}{\delta^n} ,$$

that is, $\|A\| < \frac{\varepsilon}{\delta^n} < \infty$. $\square$

We have seen above (Example 1) that not every linear transformation has a finite norm, that is, a linear transformation is not always continuous. We have also pointed out that continuity can fail for a linear transformation only when the transformation is defined on an infinite-dimensional space.

From here on $\mathcal{L}(X_1, \dots, X_n; Y)$ will denote the set of c o n t i n u o u s multilinear transformations mapping the direct product of the normed vector spaces $X_1, \dots, X_n$ into the normed vector space Y.

In particular, $\mathcal{L}(X; Y)$ is the set of continuous linear transformations from X into Y.

In the set $\mathcal{L}(X_1, \dots, X_n; Y)$ we introduce a natural vector-space structure:

$$(A + B)(x_1, \dots, x_n) := A(x_1, \dots, x_n) + B(x_1, \dots, x_n)$$

and

$$(\lambda A)(x_1, \dots, x_n) := \lambda A(x_1, \dots, x_n) .$$

It is obvious that if $A, B \in \mathcal{L}(X_1, \dots, X_n; Y)$, then $(A + B) \in \mathcal{L}(X_1, \dots, X_n; Y)$ and $(\lambda A) \in \mathcal{L}(X_1, \dots, X_n; Y)$.

Thus $\mathcal{L}(X_1, \dots, X_n; Y)$ can be regarded as a vector space.

Proposition 2. *The norm of a multilinear transformation is a norm in the vector space $\mathcal{L}(X_1, \dots, X_n; Y)$ of continuous multilinear transformations.*

Proof. We observe first of all that by Proposition 1 the nonnegative number $\|A\| < \infty$ is defined for every transformation $A \in \mathcal{L}(X_1, \dots, X_n; Y)$.

Inequality (10.27) shows that

$$\|A\| = 0 \Leftrightarrow A = 0 .$$

Next, by definition of the norm of a multilinear transformation

$$\|\lambda A\| = \sup_{\substack{x_1, \dots, x_n \\ x_i \neq 0}} \frac{|(\lambda A)(x_1, \dots, x_n)|}{|x_1| \cdot \ldots \cdot |x_n|} = \sup_{\substack{x_1, \dots, x_n \\ x_i \neq 0}} \frac{|\lambda| \, |A(x_1, \dots, x_n)|}{|x_1| \cdot \ldots \cdot |x_n|} = |\lambda| \, \|A\| .$$

Finally, if A and B are elements of the space $\mathcal{L}(X_1, \dots, X_n; Y)$, then

$$\|A + B\| = \sup_{\substack{x_1, \dots, x_n \\ x_i \neq 0}} \frac{|(A + B)(x_1, \dots, x_n)|}{|x_1| \cdot \ldots \cdot |x_n|} = \sup_{\substack{x_1, \dots, x_n \\ x_i \neq 0}} \frac{|A(x_1, \dots, x_n) + B(x_1, \dots, x_n)|}{|x_1| \cdot \ldots \cdot |x_n|} \leq$$

$$\leq \sup_{\substack{x_1, \dots, x_n \\ x_i \neq 0}} \frac{|A(x_1, \dots, x_n)|}{|x_1| \cdot \ldots \cdot |x_n|} + \sup_{\substack{x_1, \dots, x_n \\ x_i \neq 0}} \frac{|B(x_1, \dots, x_n)|}{|x_1| \cdot \ldots \cdot |x_n|} = \|A\| + \|B\| . \quad \square$$

From now on when we use the symbol $\mathcal{L}(X_1, \dots, X_n; Y)$ we shall have in mind the vector space of *continuous n-linear transformations* normed by this *transformation norm*. In particular $\mathcal{L}(X, Y)$ is the normed space of continuous linear transformations from X into Y.

We now prove the following useful supplement to Proposition 2.

Supplement. *If X, Y, and Z are normed spaces and $A \in \mathcal{L}(X;Y)$ and $B \in \mathcal{L}(Y;Z)$, then*

$$\|B \circ A\| \le \|B\| \cdot \|A\| .$$

Proof. Indeed,

$$\|B \circ A\| = \sup_{x \neq 0} \frac{|(B \circ A)x|}{|x|} \le \sup_{x \neq 0} \frac{\|B\| \, |Ax|}{|x|} =$$
$$= \|B\| \sup_{x \neq 0} \frac{|Ax|}{|x|} = \|B\| \cdot \|A\| . \ \square$$

Proposition 3. *If Y is a complete normed space, then $\mathcal{L}(X_1, \dots, X_n; Y)$ is also a complete normed space.*

Proof. We shall carry out the proof for the space $\mathcal{L}(X;Y)$ of continuous linear transformations. The general case, as will be clear from the reasoning below, differs only in requiring a more cumbersome notation.

Let $A_1, A_2, \dots, A_n \dots$ be a Cauchy sequence in $\mathcal{L}(X;Y)$. Since for any $x \in X$ we have

$$|A_m x - A_n x| = \big|(A_m - A_n)x\big| \le \|A_m - A_n\| \, |x| ,$$

it is clear that for any $x \in X$ the sequence $A_1 x$, $A_2 x$, $\dots$, $A_n x$, $\dots$ is a Cauchy sequence in Y. Since Y is complete, it has a limit in Y, which we denote by Ax.

Thus,

$$Ax := \lim_{n \to \infty} A_n x .$$

We shall show that $A : X \to Y$ is a continuous linear transformation.

The linearity of A follows from the relations

$$\lim_{n \to \infty} A_n(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n \to \infty} (\lambda_1 A_n x_1 + \lambda_2 A_n x_2) =$$
$$\lambda_1 \lim_{n \to \infty} A_n x_1 + \lambda_2 \lim_{n \to \infty} A_n x_2 .$$

Next, for any fixed $\varepsilon > 0$ and sufficiently large values of $m, n \in \mathbb{N}$ we have $\|A_m - A_n\| < \varepsilon$, and therefore

$$|A_m x - A_n x| \le \varepsilon |x|$$

at each vector $x \in X$. Letting m tend to infinity in this last relation and using the continuity of the norm of a vector, we obtain

$$|Ax - A_n x| \le \varepsilon |x| .$$

Thus $\|A - A_n\| \le \varepsilon$, and since $A = A_n + (A - A_n)$, we conclude that

$$\|A\| \le \|A_n\| + \varepsilon .$$

Consequently, we have shown that $A \in \mathcal{L}(X;Y)$ and $\|A - A_n\| \to 0$ as $n \to \infty$, that is, $A = \lim\limits_{n\to\infty} A_n$ in the sense of the norm of the space $\mathcal{L}(X;Y)$. □

In conclusion, we make one special remark relating to the space of multilinear transformations, which we shall need when studying higher-order differentials.

Proposition 4. *For each* $m \in \{1, \dots, n\}$ *there is a bijection between the spaces*

$$\mathcal{L}(X_1, \dots, X_m; \mathcal{L}(X_{m+1}, \dots, X_n; Y)) \text{ and } \mathcal{L}(X_1, \dots, X_n; Y)$$

that preserves the vector-space structure and the norm.

Proof. We shall exhibit this isomorphism.

Let $\mathfrak{B} \in \mathcal{L}(X_1, \dots, X_m; \mathcal{L}(X_{m+1}, \dots, X_n; Y))$, that is, $\mathfrak{B}(x_1, \dots, x_m) \in \mathcal{L}(X_{m+1}, \dots, X_n; Y)$.

We set

$$A(x_1, \dots, x_n) := \mathfrak{B}(x_1, \dots, x_m)(x_{m+1}, \dots, x_n) . \tag{10.30}$$

Then

$$\|\mathfrak{B}\| = \sup_{\substack{x_1,\dots,x_m \\ x_i \ne 0}} \frac{\|\mathfrak{B}(x_1, \dots, x_m)\|}{|x_1| \cdot \ldots \cdot |x_m|} = \sup_{\substack{x_1,\dots,x_m \\ x_i \ne 0}} \frac{\sup\limits_{\substack{x_{m+1},\dots,x_n \\ x_j \ne 0}} \dfrac{|\mathfrak{B}(x_1, \dots, x_m)(x_{m+1}, \dots, x_n)|}{|x_{m+1}| \cdot \ldots \cdot |x_n|}}{|x_1| \cdot \ldots \cdot |x_m|} = \sup_{\substack{x_1,\dots,x_n \\ x_k \ne 0}} \frac{|A(x_1, \dots, x_n)|}{|x_1| \cdot \ldots \cdot |x_n|} = \|A\| .$$

We leave to the reader the verification that relation (10.30) defines an isomorphism of these vector spaces. □

Applying Proposition 4 n times, we find that the space

$$\mathcal{L}(X_1; \mathcal{L}(X_2; \dots; \mathcal{L}(X_n; Y)) \cdots)$$

is isomorphic to the space $\mathcal{L}(X_1, \dots, X_n; Y)$ of n-linear transformations.

10.2.4 Problems and Exercises

1. a) Prove that if $A : X \to Y$ is a linear transformation from the normed space X into the normed space Y and X is finite-dimensional, then A is a continuous operator.

b) Prove the proposition analogous to that stated in a) for a multilinear operator.

2. Two normed vector spaces are *isomorphic* if there exists an isomorphism between them (as vector spaces) that is continuous together with its inverse transformation.

a) Show that normed vector spaces of the same finite dimension are isomorphic.

b) Show that for the infinite-dimensional case assertion a) is generally no longer true.

c) Introduce two norms in the space $C\Big([a,b],\mathbb{R}\Big)$ in such a way that the identity mapping of $C\Big([a,b],\mathbb{R}\Big)$ is not a continuous mapping of the two resulting normed spaces.

3. Show that if a multilinear transformation of n-dimensional Euclidean space is continuous at some point, then it is continuous everywhere.

4. Let $A : E^n \to E^n$ be a linear transformation of n-dimensional Euclidean space and $A^* : E^n \to E^n$ the adjoint to this transformation.

Show the following.

a) All the eigenvalues of the operator $A \cdot A^* : E^n \to E^n$ are nonnegative.

b) If $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of the operator $A \cdot A^*$, then $\|A\| = \sqrt{\lambda_n}$.

c) If the operator A has an inverse $A^{-1} : E^n \to E^n$, then $\|A^{-1}\| = \frac{1}{\sqrt{\lambda_1}}$.

d) If (a^i_j) is the matrix of the operator $A : E^n \to E^n$ in some basis, then the estimates

$$\max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} (a^i_j)^2} \leq \|A\| \leq \sqrt{\sum_{i,j=1}^{n} (a^i_j)^2} \leq \sqrt{n}\|A\|$$

hold.

5. Let $\mathbb{P}[x]$ be the vector space of polynomials in the variable x with real coefficients. We define the norm of the vector $P \in \mathbb{P}[x]$ by the formula

$$|P| = \sqrt{\int_0^1 P^2(x)\,dx}.$$

a) Is the operator $D : \mathbb{P}[x] \to \mathbb{P}[x]$ given by differentiation $(D\Big(P(x)\Big) := P'(x))$ continuous in the resulting space?

b) Find the norm of the operator $F : \mathbb{P}[x] \to \mathbb{P}[x]$ of multiplication by x, which acts according to the rule $F\Big(P(x)\Big) = x \cdot P(x)$.

6. Using the example of projection operators in $\mathbb{R}^2$, show that the inequality $\|B \circ A\| \leq \|B\| \cdot \|A\|$ may be a strict inequality.

10.3 The Differential of a Mapping

10.3.1 Mappings Differentiable at a Point

Definition 1. Let X and Y be normed spaces. A mapping $f : E \to Y$ of a set $E \subset X$ into Y is *differentiable at an interior point* $x \in E$ if there exists a continuous linear transformation $L(x) : X \to Y$ such that

$$f(x+h) - f(x) = L(x)h + \alpha(x;h) , \tag{10.31}$$

where $\alpha(x;h) = o(h)$ as $h \to 0$, $x+h \in E$.[1]

Definition 2. The function $L(x) \in \mathcal{L}(X;Y)$ that is linear with respect to h and satisfies relation (10.31) is called the *differential*, the *tangent mapping*, or the *derivative of the mapping* $f : E \to Y$ *at the point* x.

As before, we shall denote $L(x)$ by $\mathrm{d}f(x)$, $Df(x)$, or $f'(x)$.

We thus see that the general definition of differentiability of a mapping at a point is a nearly verbatim repetition of the one already familiar to us from Sect. 8.2, where it was considered in the case $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$. For that reason, from now on we shall allow ourselves to use such concepts introduced there as *increment of a function*, *increment of the argument*, and *tangent space at a point* without repeating the explanations, preserving the corresponding notation.

We shall, however, verify the following proposition in general form.

Proposition 1. *If a mapping $f : E \to Y$ is differentiable at an interior point x of a set $E \subset X$, its differential $L(x)$ at that point is uniquely determined.*

Proof. Thus we are verifying the uniqueness of the differential.

Let $L_1(x)$ and $L_2(x)$ be linear mappings satisfying relation (10.31), that is

$$\begin{aligned} f(x+h) - f(x) - L_1(x)h &= \alpha_1(x;h) , \\ f(x+h) - f(x) - L_2(x)h &= \alpha_2(x;h) , \end{aligned} \tag{10.32}$$

where $\alpha_i(x;h) = o(h)$ as $h \to 0$, $x+h \in E$, $i = 1, 2$.

Then, setting $L(x) = L_2(x) - L_1(x)$ and $\alpha(x;h) = \alpha_2(x;h) - \alpha_1(x;h)$ and subtracting the second equality in (10.32) from the first, we obtain

$$L(x)h = \alpha(x;h) .$$

Here $L(x)$ is a mapping that is linear with respect to h, and $\alpha(x;h) = o(h)$ as $h \to 0$, $x+h \in E$. Taking an auxiliary numerical parameter λ, we can now

[1] The notation "$\alpha(x;h) = o(h)$ as $h \to 0$, $x+h \in E$", of course, means that $\lim\limits_{h\to 0,\, x+h\in E} |\alpha(x;h)|_Y \cdot |h|_X^{-1} = 0$.

write

$$|L(x)h| = \frac{|L(x)(\lambda h)|}{|\lambda|} = \frac{|\alpha(x;\lambda h)|}{|\lambda h|}|h| \to 0 \text{ as } \lambda \to 0 .$$

Thus $L(x)h = 0$ for any $h \neq 0$ (we recall that x is an interior point of E). Since $L(x)0 = 0$, we have shown that $L_1(x)h = L_2(x)h$ for every value of h. □

If E is an open subset of X and $f : E \to Y$ is a mapping that is differentiable at each point $x \in E$, that is, *differentiable on* E, by the uniqueness of the differential of a mapping at a point, which was just proved, a function $E \ni x \mapsto f'(x) \in \mathcal{L}(X;Y)$ arises on the set E, which we denote $f' : E \to \mathcal{L}(X;Y)$. This mapping is called the *derivative of* f, or the *derivative mapping* relative to the original mapping $f : E \to Y$. The value $f'(x)$ of this function at an individual point $x \in E$ is the continuous linear transformation $f'(x) \in \mathcal{L}(X;Y)$ that is the differential or derivative of the function f at the particular point $x \in E$.

We note that by the requirement of continuity of the linear mapping $L(x)$ Eq. (10.31) implies that a mapping that is differentiable at a point is necessarily continuous at that point.

The converse is of course not true, as we have seen in the case of numerical functions.

We now make one more important remark.

Remark. If the condition for differentiability of the mapping f at some point a is written as

$$f(x) - f(a) = L(x)(x-a) + \alpha(a;x) ,$$

where $\alpha(a;x) = o(x-a)$ as $x \to a$, it becomes clear that Definition 1 actually applies to a mapping $f : A \to B$ of any affine spaces (A,X) and (B,Y) whose vector spaces X and Y are normed. Such affine spaces, called *normed affine spaces*, are frequently encountered, so that it is useful to keep this remark in mind when using the differential calculus.

Everything that follows, unless specifically stated otherwise, applies equally to both normed vector spaces and normed affine spaces, and we use the notation for vector spaces only for the sake of simplicity.

10.3.2 The General Rules for Differentiation

The following general properties of the operation of differentiation follow from Definition 1. In the statements below X, Y, and Z are normed spaces and U and V open sets in X and Y respectively.

a. Linearity of Differentiation *If the mappings* $f_i : U \to Y$, $i = 1, 2$, *are differentiable at a point* $x \in U$, *a linear combination of them* $(\lambda_1 f_1 + \lambda_2 f_2) : U \to Y$ *is also differentiable at* x, *and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'(x) = \lambda_1 f_1'(x) + \lambda_2 f_2'(x) .$$

Thus the differential of a linear combination of mappings is the corresponding linear combination of their differentials.

b. Differentiation of a Composition of Mappings (Chain Rule) *If the mapping* $f : U \to V$ *is differentiable at a point* $x \in U \subset X$, *and the mapping* $g : V \to Z$ *is differentiable at* $f(x) = y \in V \subset Y$, *then the composition* $g \circ f$ *of these mappings is differentiable at* x, *and*

$$(g \circ f)'(x) = g'\big(f(x)\big) \circ f'(x) .$$

Thus, the differential of a composition is the composition of the differentials.

c. Differentiation of the Inverse of a Mapping *Let* $f : U \to Y$ *be a mapping that is continuous at* $x \in U \subset X$ *and has an inverse* $f^{-1} : V \to X$ *that is defined in a neighborhood of* $y = f(x)$ *and continuous at that point.*

If the mapping f *is differentiable at* x *and its tangent mapping* $f'(x) \in \mathcal{L}(X; Y)$ *has a continuous inverse* $\big[f'(x)\big]^{-1} \in \mathcal{L}(Y; X)$, *then the mapping* f^{-1} *is differentiable at* $y = f(x)$ *and*

$$\big[f^{-1}\big]'(f(x)) = \big[f'(x)\big]^{-1} .$$

Thus, the differential of an inverse mapping is the linear mapping inverse to the differential of the original mapping at the corresponding point.

We omit the proofs of a, b, and c, since they are analogous to the proofs given in Sect. 8.3 for the case $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$.

10.3.3 Some Examples

Example 1. If $f : U \to Y$ is a constant mapping of a neighborhood $U = U(x) \subset X$ of the point x, that is, $f(U) = y_0 \in Y$, then $f'(x) = 0 \in \mathcal{L}(X; Y)$.

Proof. Indeed, in this case it is obvious that

$$f(x + h) - f(x) - 0h = y_0 - y_0 - 0 = 0 = o(h) . \ \square$$

Example 2. If the mapping $f : X \to Y$ is a continuous linear mapping of a normed vector space X into a normed vector space Y, then $f'(x) = f \in \mathcal{L}(X; Y)$ at any point $x \in A$.

Proof. Indeed,

$$f(x+h) - f(x) - fh = fx + fh - fx - fh = 0 \,. \quad \square$$

We remark that strictly speaking $f'(x) \in \mathcal{L}(TX_x; TY_{f(x)})$ here and h is a vector of the tangent space TX_x. But parallel translation of a vector to any point $x \in X$ is defined in a vector space, and this allows us to identify the tangent space TX_x with the vector space X itself. (Similarly, in the case of an affine space (A, X) the space TA_a of vectors "attached" to the point $a \in A$ can be identified with the vector space X of the given affine space.) Consequently, after choosing a basis in X, we can extend it to all the tangent spaces TX_x. This means that if, for example, $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, and the mapping $f \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ is given by the matrix (a_i^j), then at every point $x \in \mathbb{R}^m$ the tangent mapping $f'(x) : T\mathbb{R}^m_x \to T\mathbb{R}^n_{f(x)}$ will be given by the same matrix.

In particular, for a linear mapping $x \overset{f}{\longmapsto} ax = y$ from $\mathbb{R}$ to $\mathbb{R}$ with $x \in \mathbb{R}$ and $h \in T\mathbb{R}_x \sim \mathbb{R}$, we obtain the corresponding mapping $T\mathbb{R}_x \ni h \overset{f'}{\longmapsto} ah \in T\mathbb{R}_{f(x)}$.

Taking account of these conventions, we can provisionally state the result of Example 2 as follows: The mapping $f' : X \to Y$ that is the derivative of a linear mapping $f : X \to Y$ of normed spaces is constant, and $f'(x) = f$ at each point $x \in X$.

Example 3. From the chain rule for differentiating a composition of mappings and the result of Example 2 one can conclude that if $f : U \to Y$ is a mapping of a neighborhood $U = U(x) \subset X$ of the point $x \in X$ and is differentiable at x, while $A \in \mathcal{L}(Y; Z)$, then

$$(A \circ f)'(x) = A \circ f'(x) \,.$$

For numerical functions, when $Y = Z = \mathbb{R}$, this is simply the familiar possibility of moving a constant factor outside the differentiation sign.

Example 4. Suppose once again that $U = U(x)$ is a neighborhood of the point x in a normed space X, and let

$$f : U \to Y = Y_1 \times \cdots \times Y_n$$

be a mapping of U into the direct product of the normed spaces $Y_1, \dots, Y_n$.

Defining such a mapping is equivalent to defining the n mappings $f_i : U \to Y_i$, $i = 1, \dots, n$, connected with f by the relation

$$x \mapsto f(x) = y = (y_1, \dots, y_n) = \big(f_1(x), \dots, f_n(x)\big) \,,$$

which holds at every point of U.

If we now take account of the fact that in formula (10.31) we have

$$f(x+h)-f(x)=\left(f_1(x+h)-f_1(x),\dots,f_n(x+h)-f_n(x)\right),$$
$$L(x)h=\left(L_1(x)h,\dots,L_n(x)h\right),$$
$$\alpha(x;h)=\left(\alpha_1(x;h),\dots,\alpha_n(x;h)\right),$$

then, referring to the results of Example 6 of Sect. 10.1 and Example 10 of Sect. 10.2, we can conclude that the mapping f is differentiable at x if and only if all of its components $f_i: U \to Y_i$ are differentiable at x, $i=1,\dots,n$; and when the mapping f is differentiable, we have the equality

$$f'(x)=\left(f_1'(x),\dots,f_n'(x)\right).$$

Example 5. Now let $A \in \mathcal{L}(X_1,\dots,X_n;Y)$, that is, A is a continuous n-linear transformation from the product $X_1 \times \cdots \times X_n$ of the normed vector spaces $X_1,\dots,X_n$ into the normed vector space Y.

We shall prove that the mapping

$$A: X_1\times\cdots\times X_n = X \to Y$$

is differentiable and find its differential.

Proof. Using the multilinearity of A, we find that

$$\begin{aligned}
A(x+h)-A(x)&=A(x_1+h_1,\dots,x_n+h_n)-A(x_1,\dots,x_n)=\\
&=A(x_1,\dots,x_n)+A(h_1,x_2,\dots,x_n)+\cdots+A(x_1,\dots,x_{n-1},h_n)+\\
&\quad+A(h_1,h_2,x_3,\dots,x_n)+\cdots+A(x_1,\dots,x_{n-2},h_{n-1},h_n)+\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad+A(h_1,\dots,h_n)-A(x_1,\dots,x_n).
\end{aligned}$$

Since the norm in $X=X_1\times\cdots\times X_n$ satisfies the inequalities

$$|x_i|_{X_i}\le|x|_X\le\sum_{i=1}^{n}|x_i|_{X_i},$$

and the norm $\|A\|$ of the transformation A is finite and satisfies

$$|A(\xi_1,\dots,\xi_n)|\le\|A\|\,|\xi_1|\times\cdots\times|\xi_n|,$$

we can conclude that

$$\begin{aligned}
A(x+h)-A(x)&=A(x_1+h_1,\dots,x_n+h_n)-A(x_1,\dots,x_n)=\\
&=A(h_1,x_2,\dots,x_n)+\cdots+A(x_1,\dots,x_{n-1},h_n)+\alpha(x;h),
\end{aligned}$$

where $\alpha(x;h)=o(h)$ as $h\to 0$.

But the transformation

$$L(x)h = A(h_1, x_2, \dots, x_n) + \dots + A(x_1, \dots, x_{n-1}, h_n)$$

is a continuous transformation (because A is continuous) that is linear in $h = (h_1, \dots, h_n)$.

Thus we have established that

$$A'(x)h = A'(x_1, \dots, x_n)(h_1, \dots, h_n) = \\ = A(h_1, x_2, \dots, x_n) + \dots + A(x_1, \dots, x_{n-1}, h_n) ,$$

or, more briefly,

$$\mathrm{d}A(x_1, \dots, x_n) = A(\mathrm{d}x_1, x_2, \dots, x_n) + \dots + A(x_1, \dots x_{n-1}, \mathrm{d}x_n) . \ \square$$

In particular, if:

a) $x_1 \cdot \ldots \cdot x_n$ is the product of n numerical variables, then

$$\mathrm{d}(x_1 \cdot \ldots \cdot x_n) = \mathrm{d}x_1 \cdot x_2 \cdot \ldots \cdot x_n + \dots + x_1 \cdot \ldots \cdot x_{n-1} \cdot \mathrm{d}x_n ;$$

b) $\langle x_1, x_2 \rangle$ is the inner product in E^3, then

$$\mathrm{d}\langle x_1, x_2 \rangle = \langle \mathrm{d}x_1, x_2 \rangle + \langle x_1, \mathrm{d}x_2 \rangle ;$$

c) $[x_1, x_2]$ is the vector cross product in E^3, then

$$\mathrm{d}[x_1, x_2] = [\mathrm{d}x_1, x_2] + [x_1, \mathrm{d}x_2] ;$$

d) (x_1, x_2, x_3) is the scalar triple product in E^3, then

$$\mathrm{d}(x_1, x_2, x_3) = (\mathrm{d}x_1, x_2, x_3) + (x_2, \mathrm{d}x_2, x_3) + (x_2, x_2, \mathrm{d}x_3) ;$$

e) $\det(x_1, \dots, x_n)$ is the determinant of the matrix formed from the coordinates of n vectors $x_1, \dots, x_n$ in an n-dimensional vector space X with a fixed basis, then

$$\mathrm{d}\big(\det(x_1, \dots, x_n)\big) = \det(\mathrm{d}x_1, x_2, \dots, x_n) + \dots + \det(x_1, \dots, x_{n-1}, \mathrm{d}x_n) .$$

Example 6. Let U be the subset of $\mathcal{L}(X;Y)$ consisting of the continuous linear transformations $A : X \to Y$ having continuous inverse transformations $A^{-1} : Y \to X$ (belonging to $\mathcal{L}(Y;X)$). Consider the mapping

$$U \ni A \mapsto A^{-1} \in \mathcal{L}(Y;X) ,$$

which assigns to each transformation $A \in U$ its inverse $A^{-1} \in \mathcal{L}(Y;X)$.

Proposition 2 proved below makes it possible to determine whether this mapping is differentiable.

Proposition 2. *If X is a complete space and $A \in U$, then for any $h \in \mathcal{L}(X;Y)$ such that $\|h\| < \|A^{-1}\|^{-1}$, the transformation $A+h$ also belongs to U and the following relation holds:*

$$(A+h)^{-1} = A^{-1} - A^{-1}hA^{-1} + o(h) \text{ as } h \to 0\,. \tag{10.33}$$

Proof. Since

$$(A+h)^{-1} = \bigl(A(E + A^{-1}h)\bigr)^{-1} = (E + A^{-1}h)^{-1}A^{-1}\,, \tag{10.34}$$

it suffices to find the operator $(E+A^{-1}h)^{-1}$ inverse to $(E+A^{-1}h) \in \mathcal{L}(X;X)$, where E is the identity mapping e_X of X into itself.

Let $\Delta := -A^{-1}h$. Taking account of the supplement to Proposition 2 of Sect. 10.2, we can observe that $\|\Delta\| \le \|A^{-1}\| \cdot \|h\|$, so that by the assumptions made with respect to the operator h we may assume that $\|\Delta\| \le q < 1$.

We now verify that

$$(E - \Delta)^{-1} = E + \Delta + \Delta^2 + \cdots + \Delta^n + \cdots\,, \tag{10.35}$$

where the series on the right-hand side is formed from the linear operators $\Delta^n = (\Delta \circ \cdots \circ \Delta) \in \mathcal{L}(X;X)$.

Since X is a complete normed vector space, it follows from Proposition 3 of Sect. 10.2 that the space $\mathcal{L}(X;X)$ is also complete. It then follows immediately from the relation $\|\Delta^n\| \le \|\Delta\|^n \le q^n$ and the convergence of the series $\sum\limits_{n=0}^{\infty} q^n$ for $|q| < 1$ that the series (10.35) formed from the vectors in that space converges.

The direct verification that

$$\begin{aligned}(E + \Delta + \Delta^2 + \cdots)(E - \Delta) &= \\ &= (E + \Delta + \Delta^2 + \cdots) - (\Delta + \Delta^2 + \Delta^3 + \cdots) = E\end{aligned}$$

and

$$\begin{aligned}(E - \Delta)(E + \Delta + \Delta^2 + \cdots) &= \\ &= (E + \Delta + \Delta^2 + \cdots) - (\Delta + \Delta^2 + \Delta^3 + \cdots) = E\end{aligned}$$

shows that we have indeed found $(E - \Delta)^{-1}$.

It is worth remarking that the freedom in carrying out arithmetic operations on series (rearranging the terms!) in this case is guaranteed by the absolute convergence (convergence in norm) of the series under consideration.

Comparing relations (10.34) and (10.35), we conclude that

$$\begin{aligned}(A+h)^{-1} = A^{-1} - A^{-1}hA^{-1} \;+\; &(A^{-1}h)^2A^{-1} - \cdots \\ &\cdots + (-1)^n(A^{-1}h)^nA^{-1} + \cdots\end{aligned} \tag{10.36}$$

for $\|h\| \le \|A^{-1}\|^{-1}$.

Since

$$\left\| \sum_{n=2}^{\infty}(-A^{-1}h)^n A^{-1} \right\| \leq \sum_{n=2}^{\infty} \|A^{-1}h\|^n \|A^{-1}\| \leq$$
$$\leq \|A^{-1}\|^3 \|h\|^2 \sum_{m=0}^{\infty} q^m = \frac{\|A^{-1}\|^3}{1-q}\|h\|^2 ,$$

Eq. (10.33) follows in particular from (10.36). □

Returning now to Example 6, we can say that when the space X is complete the mapping $A \overset{f}{\longmapsto} A^{-1}$ under consideration is necessarily differentiable, and

$$\mathrm{d}f(A)h = \mathrm{d}(A^{-1})h = -A^{-1}hA^{-1} .$$

In particular, this means that if A is a nonsingular square matrix and A^{-1} is its inverse, then under a perturbation of the matrix A by a matrix h whose elements are close to zero, we can write the inverse matrix $(A+h)^{-1}$ in first approximation in the following form:

$$(A+h)^{-1} \approx A^{-1} - A^{-1}hA^{-1} .$$

More precise formulas can obviously be obtained starting from Eq. (10.36).

Example 7. Let X be a complete normed vector space. The important mapping

$$\exp : \mathcal{L}(X;X) \to \mathcal{L}(X;X)$$

is defined as follows:

$$\exp A := E + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots , \tag{10.37}$$

if $A \in \mathcal{L}(X;X)$.

The series in (10.37) converges, since $\mathcal{L}(X;X)$ is a complete space and $\|\frac{1}{n!}A^n\| \leq \frac{\|A\|^n}{n!}$, while the numerical series $\sum\limits_{n=0}^{\infty} \frac{\|A\|^n}{n!}$ converges.

It is not difficult to verify that

$$\exp(A+h) = \exp A + L(A)h + o(h) \text{ as } h \to \infty , \tag{10.38}$$

where

$$L(A)h = h + \frac{1}{2!}(Ah + hA) + \frac{1}{3!}(A^2h + AhA + hA^2) + \cdots$$
$$\cdots + \frac{1}{n!}(A^{n-1}h + A^{n-2}hA + \cdots + AhA^{n-2} + hA^{n-1}) + \cdots$$

and $\|L(A)\| \leq \exp\|A\| = \mathrm{e}^{\|A\|}$, that is, $L(A) \in \mathcal{L}\big(\mathcal{L}(X;X), \mathcal{L}(X;X)\big)$.

Thus, the mapping $\mathcal{L}(X;X) \ni A \mapsto \exp A \in \mathcal{L}(X;X)$ is differentiable at every value of A.

We remark that if the operators A and h commute, that is, $Ah = hA$, then, as one can see from the expression for $L(A)h$, in this case we have $L(A)h = (\exp A)h$. In particular, for $X = \mathbb{R}$ or $X = \mathbb{C}$, instead of (10.38) we again obtain

$$\exp(A+h) = \exp A + (\exp A)h + o(h) \text{ as } h \to 0 . \tag{10.39}$$

Example 8. We shall attempt to give a mathematical description of the instantaneous angular velocity of a rigid body with a fixed point o (a top). Consider an orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ at the point o rigidly attached to the body. It is clear that the position of the body is completely characterized by the position of this orthoframe, and the triple $\{\dot{\mathbf{e}}_1, \dot{\mathbf{e}}_2, \dot{\mathbf{e}}_3\}$ of instantaneous velocities of the vectors of the frame obviously give a complete characterization of the instantaneous angular velocity of the body. The position of the frame itself $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ at time t can be given by an orthogonal matrix (α_i^j) $i, j = 1, 2, 3$ composed of the coordinates of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with respect to some fixed orthonormal frame in space. Thus, the motion of the top corresponds to a mapping $t \mapsto O(t)$ from $\mathbb{R}$ (the time axis) into the group $SO(3)$ of special orthogonal 3×3 matrices. Consequently, the angular velocity of the body, which we have agreed to describe by the triple $\{\dot{\mathbf{e}}_1, \dot{\mathbf{e}}_2, \dot{\mathbf{e}}_3\}$, is the matrix $\dot{O}(t) =: (\omega_i^j)(t) = (\dot{\alpha}_i^j)(t)$, which is the derivative of the matrix $O(t) = (\alpha_i^j)(t)$ with respect to time.

Since $O(t)$ is an orthogonal matrix, the relation

$$O(t)O^*(t) = E \tag{10.40}$$

holds at any time t, where $O^*(t)$ is the transpose of $O(t)$ and E is the identity matrix.

We remark that the product $A \cdot B$ of matrices is a bilinear function of A and B, and the derivative of the transposed matrix is obviously the transpose of the derivative of the original matrix. Differentiating (10.40) and taking account of these things, we find that

$$\dot{O}(t)O^*(t) + O(t)\dot{O}^*(t) = 0$$

or

$$\dot{O}(t) = -O(t)\dot{O}^*(t)O(t) , \tag{10.41}$$

since $O^*(t)O(t) = E$.

In particular, if we assume that the frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ coincides with the spatial frame of reference at time t, then $O(t) = E$, and it follows from (10.41) that

$$\dot{O}(t) = -\dot{O}^*(t) , \tag{10.42}$$

that is, the matrix $\dot{O}(t) =: \Omega(t) = (\omega_i^j)$ of coordinates of the vectors $\{\dot{\mathbf{e}}_1, \dot{\mathbf{e}}_2, \dot{\mathbf{e}}_3\}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ turns out to be skew-symmetric:

$$\Omega(t) = \begin{pmatrix} \omega_1^1 & \omega_1^2 & \omega_1^3 \\ \omega_2^1 & \omega_2^2 & \omega_2^3 \\ \omega_3^1 & \omega_3^2 & \omega_3^3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix} .$$

Thus the instantaneous angular velocity of a top is actually characterized by three independent parameters, as follows in our line of reasoning from relation (10.40) and is natural from the physical point of view, since the position of the frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and hence the position of the body itself, can be described by three independent parameters (in mechanics these parameters may be, for example, the Euler angles).

If we associate with each vector $\boldsymbol{\omega} = \omega^1\mathbf{e}_1 + \omega^2\mathbf{e}_2 + \omega^3\mathbf{e}_3$ in the tangent space at the point o a right-handed rotation of space with angular velocity $|\boldsymbol{\omega}|$ about the axis defined by this vector, it is not difficult to conclude from these results that at each instant of time t the body has an instantaneous angular velocity and that the velocity at that time can be adequately described by the instantaneous angular velocity vector $\boldsymbol{\omega}(t)$ (see Problem 5 below).

10.3.4 The Partial Derivatives of a Mapping

Let $U = U(a)$ be a neighborhood of the point $a \in X = X_1 \times \cdots \times X_m$ in the direct product of the normed spaces $X_1, \ldots, X_m$, and let $f : U \to Y$ be a mapping of U into the normed space V. In this case

$$y = f(x) = f(x_1, \ldots, x_m) , \tag{10.43}$$

and hence, if we fix all the variables but x_i in (10.43) by setting $x_k = a_k$ for $k \in \{1, \ldots, m\} \setminus i$, we obtain a function

$$f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_m) =: \varphi_i(x_i) , \tag{10.44}$$

defined in some neighborhood U_i of a_i in X.

Definition 3. Relative to the original mapping (10.43) the mapping $\varphi_i : U_i \to Y$ is called the *partial mapping with respect to the variable x_i at $a \in X$.*

Definition 4. If the mapping (10.44) is differentiable at $x_i = a_i$, its derivative at that point is called the *partial derivative* or *partial differential of f at a with respect to the variable x_i.*

We usually denote this partial derivative by one of the symbols

$$\partial_i f(a) , \quad D_i f(a) , \quad \frac{\partial f}{\partial x_i}(a) , \quad f'_{x_i}(a) .$$

In accordance with these definitions $D_i f(a) \in \mathcal{L}(X_i; Y)$. More precisely, $D_i f(a) \in \mathcal{L}\big(TX_i(a_i); TY(f(a))\big)$.

The differential $\mathrm{d}f(a)$ of the mapping (10.43) at the point a (if f is differentiable at that point) is often called the *total differential* in this situation in order to distinguish it from the partial differentials with respect to the individual variables.

We have already encountered all these concepts in the case of real-valued functions of m real variables, so that we shall not give a detailed discussion of them. We remark only that by repeating our earlier reasoning, taking account of Example 9 in Sect. 9.2, one can prove easily that the following proposition holds in general.

Proposition 3. *If the mapping* (10.43) *is differentiable at the point* $a = (a_1, \dots, a_m) \in X_1 \times \cdots \times X_m = X$, *it has partial derivatives with respect to each variable at that point, and the total differential and the partial differentials are related by the equation*

$$\mathrm{d}f(a)h = \partial_1 f(a)h_1 + \cdots + \partial_m f(a)h_m , \tag{10.45}$$

where $h = (h_1, \dots, h_m) \in TX_1(a_1) \times \cdots \times TX_m(a_m) = TX(a)$.

We have already shown by the example of numerical functions that the existence of partial derivatives does not in general guarantee the differentiability of the function (10.43).

10.3.5 Problems and Exercises

1. a) Let $A \in \mathcal{L}(X; X)$ be a *nilpotent operator*, that is, there exists $k \in \mathbb{N}$ such that $A^k = 0$. Show that the operator $(E - A)$ has an inverse in this case and that $(E - A)^{-1} = E + A + \cdots + A^{k-1}$.

b) Let $D : \mathbb{P}[x] \to \mathbb{P}[x]$ be the operator of differentiation on the vector space $\mathbb{P}[x]$ of polynomials. Remarking that D is a nilpotent operator, write the operator $\exp(aD)$, where $a \in \mathbb{R}$, and show that $\exp(aD)\Big(P(x)\Big) = P(x + a) =: T_a\Big(P(x)\Big)$.

c) Write the matrices of the operators $D : \mathbb{P}_n[x] \to \mathbb{P}_n[x]$ and $T_a : \mathbb{P}_n[x] \to \mathbb{P}_n[x]$ from part b) in the basis $e_i = \frac{x^{n-i}}{(n-i)!}$, $1 \le i \le n$, in the space $\mathbb{P}_n[x]$ of real polynomials of degree n in one variable.

2. a) If $A, B \in \mathcal{L}(X; X)$ and $\exists B^{-1} \in \mathcal{L}(X; X)$, then $\exp(B^{-1}AB) = B^{-1}(\exp A)B$.

b) If $AB = BA$, then $\exp(A + B) = \exp A \cdot \exp B$.

c) Verify that $\exp 0 = E$ and that $\exp A$ always has an inverse, namely $(\exp A)^{-1} = \exp(-A)$.

3. Let $A \in \mathcal{L}(X; X)$. Consider the mapping $\varphi_A : \mathbb{R} \to \mathcal{L}(X; X)$ defined by the correspondence $\mathbb{R} \ni t \mapsto \exp(tA) \in \mathcal{L}(X; X)$. Show the following.

a) The mapping φ_A is continuous.

b) φ_A is a homomorphism of $\mathbb{R}$ as an additive group into the multiplicative group of invertible operators in $\mathcal{L}(X; X)$.

4. Verify the following.

a) If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the operator $A \in \mathcal{L}(\mathbb{C}^n; \mathbb{C}^n)$, then $\exp \lambda_1, \dots, \exp \lambda_n$ are the eigenvalues of $\exp A$.

b) $\det(\exp A) = \exp(\operatorname{tr} A)$, where $\operatorname{tr} A$ is the trace of the operator $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$.

c) If $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, then $\det(\exp A) > 0$.

d) If A^* is the transpose of the matrix $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ and $\bar{A}$ is the matrix whose elements are the complex conjugates of those of A, then $(\exp A)^* = \exp A^*$ and $\overline{\exp A} = \exp \bar{A}$.

e) The matrix $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ is not of the form $\exp A$ for any 2×2 matrix A.

5. We recall that a set endowed with both a group structure and a topology is called a *topological group* or *continuous group* if the group operation is continuous. If there is a sense in which the group operation is even analytic, the topological group is called a *Lie group*.[2]

A *Lie algebra* is a vector space X with an anticommutative bilinear operation $[\ ,\ ] : X \times X \to X$ satisfying the *Jacobi identity*: $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ for any vectors $a, b, c \in X$. Lie groups and algebras are closely connected with each other, and the mapping exp plays an important role in establishing this connection (see Problem 1 above).

An example of a Lie algebra is the oriented Euclidean space E^3 with the operation of the vector cross product. For the time being we shall denote this Lie algebra by LA_1.

a) Show that the real 3×3 skew-symmetric matrices form a Lie algebra (which we denote LA_2) if the product of the matrices A and B is defined as $[A, B] = AB - BA$.

b) Show that the correspondence

$$\Omega = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix} \leftrightarrow (\omega_1, \omega_2, \omega_3) = \boldsymbol{\omega}$$

is an isomorphism of the Lie algebras LA_2 and LA_1.

c) Verify that if the skew-symmetric matrix Ω and the vector $\boldsymbol{\omega}$ correspond to each other as shown in b), then the equality $\Omega \mathbf{r} = [\boldsymbol{\omega}, \mathbf{r}]$ holds for any vector $\mathbf{r} \in E^3$, and the relation $P\Omega P^{-1} \leftrightarrow P\boldsymbol{\omega}$ holds for any matrix $P \in SO(3)$.

d) Verify that if $\mathbb{R} \ni t \mapsto O(t) \in SO(3)$ is a smooth mapping, then the matrix $\Omega(t) = O^{-1}(t)\dot{O}(t)$ is skew-symmetric.

e) Show that if $\mathbf{r}(t)$ is the radius vector of a point of a rotating top and $\Omega(t)$ is the matrix $(O^{-1}\dot{O})(t)$ found in d), then $\dot{\mathbf{r}}(t) = (\Omega \mathbf{r})(t)$.

f) Let $\mathbf{r}$ and $\boldsymbol{\omega}$ be two vectors attached at the origin of E^3. Suppose a right-handed frame has been chosen in E^3, and that the space undergoes a right-handed rotation with angular velocity $|\boldsymbol{\omega}|$ about the axis defined by $\boldsymbol{\omega}$. Show that $\dot{\mathbf{r}}(t) = [\boldsymbol{\omega}, \mathbf{r}(t)]$ in this case.

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[2] For the precise definition of a Lie group and the corresponding reference see Problem 8 in Sect. 15.2.

g) Summarize the results of d), e), and f) and exhibit the instantaneous angular velocity of the rotating top discussed in Example 8.

h) Using the result of c), verify that the velocity vector $\boldsymbol{\omega}$ is independent of the choice of the fixed orthoframe in E^3, that is, it is independent of the coordinate system.

6. Let $\mathbf{r} = \mathbf{r}(s) = \big(x^1(s), x^2(s), x^3(s)\big)$ be the parametric equations of a smooth curve in E^3, the parameter being arc length along the curve (the *natural parametrization of the curve*).

a) Show that the vector $\mathbf{e}_1(s) = \frac{d\mathbf{r}}{ds}(s)$ tangent to the curve has unit length.

b) The vector $\frac{d\mathbf{e}_1}{ds}(s) = \frac{d^2\mathbf{r}}{ds^2}(s)$ is orthogonal to $\mathbf{e}_1$. Let $\mathbf{e}_2(s)$ be the unit vector formed from $\frac{d\mathbf{e}_1}{ds}(s)$. The coefficient $k(s)$ in the equality $\frac{d\mathbf{e}_1}{ds}(s) = k(s)\mathbf{e}_2(s)$ is called the *curvature* of the curve at the corresponding point.

c) By constructing the vector $\mathbf{e}_3(s) = [\mathbf{e}_1(s), \mathbf{e}_2(s)]$ we obtain a frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ at each point, called the *Frenet* frame[3] or *companion trihedral* of the curve. Verify the following Frenet formulas:

$$\begin{aligned}
\frac{d\mathbf{e}_1}{ds}(s) &= \phantom{-k(s)\mathbf{e}_1(s)} \quad k(s)\mathbf{e}_2(s)\,, \\
\frac{d\mathbf{e}_2}{ds}(s) &= -k(s)\mathbf{e}_1(s) \qquad\qquad\qquad \varkappa(s)\mathbf{e}_3(s)\,, \\
\frac{d\mathbf{e}_3}{ds}(s) &= \phantom{-k(s)\mathbf{e}_1(s)} \; -\varkappa(s)\mathbf{e}_2(s)\,.
\end{aligned}$$

Explain the geometric meaning of the coefficient $\varkappa(s)$ called the *torsion* of the curve at the corresponding point.

10.4 The Finite-increment Theorem and some Examples of its Use

10.4.1 The Finite-increment Theorem

In our study of numerical functions of several variables in Subsect. 5.3.1 we proved the finite-increment theorem for them and discussed in detail various aspects of this important theorem of analysis. In the present section the finite-increment theorem will be proved in its general form. So that its meaning will be fully obvious, we advise the reader to recall the discussion in that subsection and also to pay attention to the geometric meaning of the norm of a linear operator (see Subsect. 10.2.2).

Theorem 1. (The finite-increment theorem). *Let $f : U \to Y$ be a continuous mapping of an open set U of a normed space X into a normed space Y.*

If the closed interval $[x, x+h] = \{\xi \in X \mid \xi = x + \theta h,\ 0 \le \theta \le 1\}$ is contained in U and the mapping f is differentiable at all points of the open interval $]x, x+h[= \{\xi \in X \mid \xi = x + \theta h,\ 0 < \theta < 1\}$, then the following

[3] J. F. Frenet (1816–1900) – French mathematician.

estimate holds:

$$|f(x+h) - f(x)|_Y \le \sup_{\xi \in]x, x+h[} \|f'(\xi)\|_{\mathcal{L}(X;Y)} |h|_X . \tag{10.46}$$

Proof. We remark first of all that if we could prove the inequality

$$|f(x'') - f(x')| \le \sup_{\xi \in [x', x'']} \|f'(\xi)\| \, |x'' - x'| \tag{10.47}$$

in which the supremum extends over the whole interval $[x', x'']$, for every closed interval $[x', x''] \subset]x, x+h[$, then, using the continuity of f and the norm together with the fact that

$$\sup_{\xi \in [x', x'']} \|f'(\xi)\| \le \sup_{\xi \in]x, x+h[} \|f'(\xi)\| ,$$

we would obtain inequality (10.46) in the limit as $x' \to x$ and $x'' \to x + h$.

Thus, it suffices to prove that

$$|f(x+h) - f(x)| \le M|h| , \tag{10.48}$$

where $M = \sup\limits_{0 \le \theta \le 1} \|f'(x + \theta h)\|$ and the function f is assumed differentiable on the entire closed interval $[x, x+h]$.

The very simple computation

$$\begin{aligned} |f(x_3) - f(x_1)| &\le |f(x_3) - f(x_2)| + |f(x_2) - f(x_1)| \le \\ &\le M|x_3 - x_2| + M|x_2 - x_1| = M\big(|x_3 - x_2| + |x_2 - x_1|\big) = \\ &= M|x_3 - x_1| , \end{aligned}$$

which uses only the triangle inequality and the properties of a closed interval, shows that if an inequality of the form (10.48) holds on the portions $[x_1, x_2]$ and $[x_2, x_3]$ of the closed interval $[x_1, x_3]$, then it also holds on $[x_1, x_3]$.

Hence, if estimate (10.48) fails for the closed interval $[x, x+h]$, then by successive bisections, one can obtain a sequence of closed intervals $[a_k, b_k] \subset]x, x+h[$ contracting to some point $x_0 \in [x, x+h]$ such that (10.48) fails on each interval $[a_k, b_k]$. Since $x_0 \in [a_k, b_k]$, consideration of the closed intervals $[a_k, x_0]$ and $[x_0, b_k]$ enables us to assume that we have found a sequence of closed intervals of the form $[x_0, x_0 + h_k] \subset [x, x+h]$, where $h_k \to 0$ as $k \to \infty$ on which

$$|f(x_0 + h_k) - f(x_0)| > M|h_k| . \tag{10.49}$$

If we prove (10.48) with M replaced by $M + \varepsilon$, where ε is any positive number, we will still obtain (10.48) as $\varepsilon \to 0$, and hence we can also replace (10.49) by

$$|f(x_0 + h_k) - f(x_0)| > (M + \varepsilon)|h_k| \tag{10.49'}$$

and we can now show that this is incompatible with the assumption that f is differentiable at x_0.

Indeed, by the assumption that f is differentiable,

$$|f(x_0+h_k)-f(x_0)| = |f'(x_0)h_k + o(h_k)| \leq \\ \leq \|f'(x_0)\|\,|h_k| + o(|h_k|) \leq (M+\varepsilon)|h_k|$$

as $h_k \to 0$. □

The finite-increment theorem has the following useful, purely technical corollary.

Corollary. *If $A \in \mathcal{L}(X;Y)$, that is, A is a continuous linear mapping of the normed space X into the normed space Y and $f: U \to Y$ is a mapping satisfying the hypotheses of the finite-increment theorem, then*

$$|f(x+h) - f(x) - Ah| \leq \sup_{\xi \in]x,x+h[} \|f'(\xi) - A\|\,|h|\ .$$

Proof. For the proof it suffices to apply the finite-increment theorem to the mapping

$$t \mapsto F(t) = f(x+th) - Ath$$

of the unit interval $[0,1] \subset \mathbb{R}$ into Y, since

$$F(1) - F(0) = f(x+h) - f(x) - Ah\ ,$$
$$F'(\theta) = f'(x+\theta h)h - Ah \text{ for } 0 < \theta < 1\ ,$$
$$\|F'(\theta)\| \leq \|f'(x+\theta h) - A\|\,|h|\ ,$$
$$\sup_{0<\theta<1} \|F'(\theta)\| \leq \sup_{\xi \in]x,x+h[} \|f'(\xi) - A\|\,|h|\ .\ \square$$

Remark. As can be seen from the proof of Theorem 1, in its hypotheses there is no need to require that f be differentiable as a mapping $f: U \to Y$; it suffices that its restriction to the closed interval $[x, x+h]$ be a continuous mapping of that interval and differentiable at the points of the open interval $]x, x+h[$.

This remark applies equally to the corollary of the finite-increment theorem just proved.

10.4.2 Some Applications of the Finite-increment Theorem

a. Continuously Differentiable Mappings Let

$$f: U \to Y \tag{10.50}$$

be a mapping of an open subset U of a normed vector space X into a normed space Y. If f is differentiable at each point $x \in U$, then, assigning to the point x the mapping $f'(x) \in \mathcal{L}(X;Y)$ tangent to f at that point, we obtain the derivative mapping

$$f' : U \to \mathcal{L}(X;Y) . \tag{10.51}$$

Since the space $\mathcal{L}(X;Y)$ of continuous linear transformations from X into Y is, as we know, a normed space (with the transformation norm), it makes sense to speak of the continuity of the mapping (10.51).

Definition. When the derivative mapping (10.51) is continuous in U, the mapping (10.50), in complete agreement with our earlier terminology, will be said to be *continuously differentiable*.

As before, the set of continuously differentiable mappings of type (10.50) will be denoted by the symbol $C^{(1)}(U,Y)$, or more briefly, $C^{(1)}(U)$, if it is clear from the context what the range of the mapping is.

Thus, by definition

$$f \in C^{(1)}(U,Y) \Leftrightarrow f' \in C\big(U, \mathcal{L}(X;Y)\big) .$$

Let us see what continuous differentiability of a mapping means in different particular cases.

Example 1. Consider the familiar situation when $X = Y = \mathbb{R}$, and hence $f : U \to \mathbb{R}$ is a real-valued function of a real argument. Since any linear mapping $A \in \mathcal{L}(\mathbb{R};\mathbb{R})$ reduces to multiplication by some number $a \in \mathbb{R}$, that is, $Ah = ah$ and obviously $\|A\| = |a|$, we find that $f'(x)h = a(x)h$, where $a(x)$ is the numerical derivative of the function f at the point x.

Next, since

$$\begin{aligned}\big(f'(x+\delta) - f'(x)\big)h &= f'(x+\delta)h - f'(x)h = \\ &= a(x+\delta)h - a(x)h = \big(a(x+\delta) - a(x)\big)h ,\end{aligned} \tag{10.52}$$

it follows that

$$\|f'(x+\delta) - f'(x)\| = |a(x+\delta) - a(x)|$$

and hence in this case continuous differentiability of the mapping f is equivalent to the concept of a continuously differentiable numerical function (of class $C^{(1)}(U,\mathbb{R})$) studied earlier.

Example 2. This time suppose that X is the direct product $X_1 \times \cdots \times X_n$ of normed spaces. In this case the mapping (10.50) is a function $f(x) = f(x_1,\dots,x_m)$ of m variables $x_i \in X_i$, $i = 1,\dots,m$, with values in Y.

If the mapping f is differentiable at $x \in U$, its differential $\mathrm{d}f(x)$ at that point is an element of the space $\mathcal{L}(X_1 \times \cdots \times X_m = X;Y)$.

The action of $\mathrm{d}f(x)$ on a vector $h = (h_1,\dots,h_m)$, by formula (10.45), can be represented as

$$\mathrm{d}f(x)h = \partial_1 f(x)h_1 + \cdots + \partial_m f(x)h_m ,$$

where $\partial_i f(x) : X_i \to Y$, $i = 1, \dots, m$, are the partial derivatives of the mapping f at the point x under consideration.

Next,

$$\big(\mathrm{d}f(x+\delta) - \mathrm{d}f(x)\big)h = \sum_{i=1}^{m} \big(\partial_i f(x+\delta) - \partial_i f(x)\big)h_i \,. \tag{10.53}$$

But by the properties of the standard norm in the direct product of normed spaces (see Example 6 in Subsect. 10.1.2) and the definition of the norm of a transformation, we find that

$$\|\partial_i f(x+\delta) - \partial_i f(x)\|_{\mathcal{L}(X_i;Y)} \le \|\mathrm{d}f(x+\delta) - \mathrm{d}f(x)\|_{\mathcal{L}(X;Y)} \le \\ \le \sum_{i=1}^{m} \|\partial_i f(x+\delta) - \partial_i f(x)\|_{\mathcal{L}(X_i;Y)} \,. \tag{10.54}$$

Thus in this case the differentiable mapping (10.50) is continuously differentiable in U if and only if all its partial derivatives are continuous in U.

In particular, if $X = \mathbb{R}^m$ and $Y = \mathbb{R}$, we again obtain the familiar concept of a continuously differentiable numerical function of m real variables (a function of class $C^{(1)}(U, \mathbb{R})$, where $U \subset \mathbb{R}^m$).

Remark. It is worth noting that in writing (10.52) and (10.53) we have made essential use of the canonical identification $TX_x \sim X$, which makes it possible to compare or identify vectors lying in different tangent spaces.

We shall now show that continuously differentiable mappings satisfy a Lipschitz condition.

Proposition 1. *If K is a convex compact set in a normed space X and $f \in C^{(1)}(K, Y)$, where Y is also a normed space, then the mapping $f : K \to Y$ satisfies a Lipschitz condition on K, that is, there exists a constant $M > 0$ such that the inequality*

$$|f(x_2) - f(x_1)| \le M|x_2 - x_1| \tag{10.55}$$

holds for any points $x_1, x_2 \in K$.

Proof. By hypothesis $f' : K \to \mathcal{L}(X;Y)$ is a continuous mapping of the compact set K into the metric space $\mathcal{L}(X;Y)$. Since the norm is a continuous function on a normed space with its natural metric, the mapping $x \mapsto \|f'(x)\|$, being the composition of continuous functions, is itself a continuous mapping of the compact set K into $\mathbb{R}$. But such a mapping is necessarily bounded. Let M be a constant such that $\|f'(x)\| \le M$ at any point $x \in K$. Since K is convex, for any two points $x_1 \in K$ and $x_2 \in K$ the entire interval $[x_1, x_2]$ is contained in K. Applying the finite-increment theorem to that interval, we immediately obtain relation (10.55). □

Proposition 2. *Under the hypotheses of Proposition 1 there exists a non-negative function $\omega(\delta)$ tending to 0 as $\delta \to +0$ such that*

$$|f(x+h) - f(x) - f'(x)h| \leq \omega(\delta)|h| \tag{10.56}$$

at any point $x \in K$ for $|h| < \delta$ if $x + h \in K$.

Proof. By the corollary to the finite-increment theorem we can write

$$|f(x+h) - f(x) - f'(x)h| \leq \sup_{0<\theta<1} \|f'(x+\theta h) - f'(x)\| \, |h|$$

and, setting

$$\omega(\delta) = \sup_{\substack{x_1, x_2 \in K \\ |x_1 - x_2| < \delta}} \|f'(x_2) - f'(x_1)\| ,$$

we obtain (10.56) in view of the uniform continuity of the function $x \mapsto f'(x)$, which is continuous on the compact set K. □

b. A Sufficient Condition for Differentiability We shall now show that by using the general finite-increment theorem, we can obtain a general sufficient condition for differentiability of a mapping in terms of its partial derivatives.

Theorem 2. *Let U be a neighborhood of the point x in a normed space $X = X_1 \times \cdots \times X_m$, which is the direct product of the normed spaces $X_1 \times \cdots \times X_m$, and let $f : U \to Y$ be a mapping of U into a normed space Y. If the mapping f has partial derivatives with respect to all its variables in U, then it is differentiable at the point x if the partial derivatives are all continuous at that point.*

Proof. To simplify the writing we carry out the proof for the case $m = 2$. We verify immediately that the mapping

$$Lh = \partial_1 f(x) h_1 + \partial_2 f(x) h_2 ,$$

which is linear in $h = (h_1, h_2)$, is the total differential of f at x.

Making the elementary transformations

$$\begin{aligned}
f(x+h) &- f(x) - Lh = \\
&= f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) - \partial_1 f(x) h_1 - \partial_2 f(x) h_2 = \\
&= f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2 + h_2) - \partial_1 f(x_1, x_2) h_1 + \\
&\qquad + f(x_1, x_2 + h_2) - f(x_1, x_2) - \partial_2 f(x_1, x_2) h_2 ,
\end{aligned}$$

by the corollary to Theorem 1 we obtain

$$\begin{aligned}
|f(x_1 + h_1, x_2 + h_2) &- f(x_1, x_2) - \partial_1 f(x_1, x_2) h_1 - \partial_2 f(x_1, x_2) h_2| \leq \\
&\leq \sup_{0<\theta_1<1} \|\partial_1 f(x_1 + \theta_1 h_1, x_2 + h_2) - \partial_1 f(x_1, x_2)\| \, |h_1| + \\
&\qquad + \sup_{0<\theta_2<1} \|\partial_2 f(x_1, x_2 + \theta_2 h_2) - \partial_2 f(x_1, x_2)\| \, |h_2| .
\end{aligned} \tag{10.57}$$

Since $\max\{|h_1|, |h_2\} \le |h|$, it follows obviously from the continuity of the partial derivatives $\partial_1 f$ and $\partial_2 f$ at the point $x = (x_1, x_2)$ that the right-hand side of inequality (10.57) is $o(h)$ as $h = (h_1, h_2) \to 0$. □

Corollary. *A mapping $f : U \to Y$ of an open subset U of the normed space $X = X_1 \times \cdots \times X_m$ into a normed space Y is continuously differentiable if and only if all the partial derivatives of the mapping f are continuous.*

Proof. We have shown in Example 2 that when the mapping $f : U \to Y$ is differentiable, it is continuously differentiable if and only if its partial derivatives are continuous.

We now see that if the partial derivatives are continuous, then the mapping f is automatically differentiable, and hence (by Example 2) also continuously differentiable. □

10.4.3 Problems and Exercises

1. Let $f : I \to Y$ be a continuous mapping of the closed interval $I = [0, 1] \subset \mathbb{R}$ into a normed space Y and $g : I \to \mathbb{R}$ a continuous real-valued function on I. Show that if f and g are differentiable in the open interval $]0, 1[$ and the relation $\|f'(t)\| \le g'(t)$ holds at points of this interval, then the inequality $|f(1) - f(0)| \le g(1) - g(0)$ also holds.

2. a) Let $f : I \to Y$ be a continuously differentiable mapping of the closed interval $I = [0, 1] \subset \mathbb{R}$ into a normed space Y. It defines a smooth path in Y. Define the length of that path.

b) Recall the geometric meaning of the norm of the tangent mapping and give an upper bound for the length of the path considered in a).

c) Give a geometric interpretation of the finite-increment theorem.

3. Let $f : U \to Y$ be a continuous mapping of a neighborhood U of the point a in a normed space X into a normed space Y. Show that if f is differentiable in $U \setminus a$ and $f'(x)$ has a limit $L \in \mathcal{L}(X; Y)$ as $x \to a$, then the mapping f is differentiable at a and $f'(a) = L$.

4. a) Let U be an open convex subset of a normed space X and $f : U \to Y$ a mapping of U into a normed space Y. Show that if $f'(x) \equiv 0$ on U, then the mapping f is constant.

b) Generalize the assertion of a) to the case of an arbitrary domain U (that is, when U is an open connected subset of X).

c) The partial derivative $\frac{\partial f}{\partial y}$ of a smooth function $f : D \to \mathbb{R}$ defined in a domain $D \subset \mathbb{R}^2$ of the xy-plane is identically zero. Is it true that f is then independent of y in this domain? For which domains D is this true?

10.5 Higher-order Derivatives

10.5.1 Definition of the nth Differential

Let U be an open set in a normed space X and

$$f : U \to Y \tag{10.58}$$

a mapping of U into a normed space Y.

If the mapping (10.58) is differentiable in U, then the derivative of f, given by

$$f' : U \to \mathcal{L}(X; Y) , \tag{10.59}$$

is defined in U.

The space $\mathcal{L}(X; Y) =: Y_1$ is a normed space relative to which the mapping (10.59) has the form (10.58), that is, $f' : U \to Y_1$, and it makes sense to speak of differentiability for it.

If the mapping (10.59) is differentiable, its derivative

$$(f')' : U \to \mathcal{L}(X; Y_1) = \mathcal{L}(X; \mathcal{L}(X; Y))$$

is called the *second derivative* or *second differential* of f and denoted f'' or $f^{(2)}$. In general, we adopt the following inductive definition.

Definition 1. The *derivative of order* $n \in$ Nor *nth differential* of the mapping (10.58) at the point $x \in U$ is the mapping tangent to the derivative of f of order $n-1$ at that point.

If the derivative of order $k \in \mathbb{N}$ at the point $x \in U$ is denoted $f^{(k)}(x)$, Definition 1 means that

$$f^{(n)}(x) := \left(f^{(n-1)}\right)'(x) . \tag{10.60}$$

Thus, if $f^{(n)}(x)$ is defined, then

$$f^{(n)}(x) \in \mathcal{L}(X; Y_n) = \mathcal{L}(X; \mathcal{L}(X; Y_{n-1})) = \cdots$$
$$\cdots = \mathcal{L}(X; \mathcal{L}(X; \dots; \mathcal{L}(X; Y)) \dots) .$$

Consequently, by Proposition 4 of Sect. 10.2, $f^{(n)}(x)$, the differential of order n of the mapping (10.58) at the point x can be interpreted as an element of the space $\mathcal{L}(\underbrace{X, \dots, X}_{n \text{ factors}}; Y)$ of continuous n-linear transformations.

We note once again that the tangent mapping $f'(x) : TX_x \to TY_{f(x)}$ is a mapping of tangent spaces, each of which, because of the affine or vector-space structure of the spaces being mapped, we have identified with the corresponding vector space and said on that basis that $f'(x) \in \mathcal{L}(X; Y)$. It is this device of regarding elements $f'(x_1) \in \mathcal{L}(TX_{x_1}; TY_{f(x_1)})$ and

$f'(x_2) \in \mathcal{L}(TX_{x_2}, TY_{f(x_2)})$, which lie in different spaces, as vectors in the same space $\mathcal{L}(X;Y)$ that provides the basis for defining higher-order differentials of mappings of normed vector spaces. In the case of an affine or vector space there is a natural connection between vectors in the different tangent spaces corresponding to different points of the original space. In the final analysis, it is this connection that makes it possible to speak of the continuous differentiability of both the mapping (10.58) and its higher-order differentials.

10.5.2 Derivative with Respect to a Vector and Computation of the Values of the nth Differential

When we are making the abstract Definition 1 specific, the concept of the derivative with respect to a vector may be used to advantage. This concept is introduced for the general mapping (10.58) just as was done earlier in the case $X = \mathbb{R}^m$, $Y = \mathbb{R}$.

Definition 2. If X and Y are normed vector spaces over the field $\mathbb{R}$, the *derivative of the mapping* (10.58) *with respect to the vector* $h \in TX_x \sim X$ *at the point* $x \in U$ is defined as the limit

$$D_h f(x) := \lim_{\mathbb{R}\ni t\to 0} \frac{f(x+th) - f(x)}{t} ,$$

provided this limit exists.

It can be verified immediately that

$$D_{\lambda h} f(x) = \lambda D_h f(x) \tag{10.61}$$

and that if the mapping f is differentiable at the point $x \in U$, it has a derivative at that point with respect to every vector; moreover

$$D_h f(x) = f'(x)h , \tag{10.62}$$

and, by the linearity of the tangent mapping,

$$D_{\lambda_1 h_1 + \lambda_2 h_2} f(x) = \lambda_1 D_{h_1} f(x) + \lambda_2 D_{h_2} f(x) . \tag{10.63}$$

It can also be seen from Definition 2 that the value $D_h f(x)$ of the derivative of the mapping $f : U \to Y$ with respect to a vector is an element of the vector space $TY_{f(x)} \sim Y$, and that if L is a continuous linear transformation from Y to a normed space Z, then

$$D_h(L \circ f)(x) = L \circ D_h f(x) . \tag{10.64}$$

We shall now try to give an interpretation to the value $f^{(n)}(h_1, \ldots, h_n)$ of the nth differential of the mapping f at the point x on the set $(h_1, \ldots, h_n)$ of vectors $h_i \in TX_x \sim X$, $i = 1, \ldots, n$.

We begin with $n = 1$. In this case, by formula (10.62)

$$f'(x)(h) = f'(x)h = D_h f(x) .$$

We now consider the case $n = 2$. Since $f^{(2)}(x) \in \mathcal{L}(X; \mathcal{L}(X; Y))$, fixing a vector $h_1 \in X$, we assign a linear transformation $\left(f^{(2)}(x)h_1\right) \in \mathcal{L}(X; Y)$ to it by the rule

$$h_1 \mapsto f^{(2)}(x)h_1 .$$

Then, after computing the value of this operator at the vector $h_2 \in X$, we obtain an element of Y:

$$f^{(2)}(x)(h_1, h_2) := \left(f^{(2)}(x)h_1\right)h_2 \in Y . \tag{10.65}$$

But

$$f^{(2)}(x)h = (f')'(x)h = D_h f'(x) ,$$

and therefore

$$f^{(2)}(x)(h_1, h_2) = \left(D_{h_1} f'(x)\right)h_2 . \tag{10.66}$$

If $A \in \mathcal{L}(X; Y)$ and $h \in X$, this pairing with Ah can be regarded not only as a mapping $h \mapsto Ah$ from X into Y, but as a mapping $A \mapsto Ah$ from $\mathcal{L}(X; Y)$ into Y, the latter mapping being linear, just like the former.

Comparing relations (10.62), (10.64), and (10.66), we can write

$$\left(D_{h_1} f'(x)\right)h_2 = D_{h_1}\left(f'(x)h_2\right) = D_{h_1} D_{h_2} f(x) .$$

Thus we finally obtain

$$f^{(2)}(x)(h_1, h_2) = D_{h_1} D_{h_2} f(x) .$$

Similarly, one can show that the relation

$$f^{(n)}(x)(h_1, \dots, h_n) := \left(\dots\left(f^{(n)}(x)h_1\right)\dots h_n\right) = D_{h_1} D_{h_2} \cdots D_{h_n} f(x) \tag{10.67}$$

holds for any $n \in \mathbb{N}$, the differentiation with respect to the vectors being carried out sequentially, starting with differentiation with respect to h_n and ending with differentiation with respect to h_1.

10.5.3 Symmetry of the Higher-order Differentials

In connection with formula (10.67), which is perfectly adequate for computation as it now stands, the question naturally arises: To what extent does the result of the computation depend on the order of differentiation?

Proposition. *If the form $f^{(n)}(x)$ is defined at the point x for the mapping (10.58), it is symmetric with respect to any pair of its arguments.*

Proof. The main element in the proof is to verify that the proposition holds in the case $n = 2$.

Let h_1 and h_2 be two arbitrary fixed vectors in the space $TX_x \sim X$. Since U is open in X, the following auxiliary function of t is defined for all values of $t \in \mathbb{R}$ sufficiently close to zero:

$$F_t(h_1, h_2) = f(x + t(h_1 + h_2)) - f(x + th_1) - f(x + th_2) + f(x) .$$

We consider also the following auxiliary function:

$$g(v) = f(x + t(h_1 + v)) - f(x + tv) ,$$

which is certainly defined for vectors v that are collinear with the vector h_2 and such that $|v| \leq |h_2|$.

We observe that

$$F_t(h_1, h_2) = g(h_2) - g(0) .$$

We further observe that, since the function $f : U \to Y$ has a second differential $f''(x)$ at the point $x \in U$, it must be differentiable at least in some neighborhood of x. We shall assume that the parameter t is sufficiently small that the arguments on the right-hand side of the equality that defines $F_t(h_1, h_2)$ lie in that neighborhood.

We now make use of these observations and the corollary of the mean-value theorem in the following computations:

$$\begin{aligned} |F_t(h_1, h_2) - t^2 f''(x)(h_1, h_2)| &= \\ &= |g(h_2) - g(0) - t^2 f''(x)(h_1, h_2)| \leq \\ &\leq \sup_{0<\theta_2<1} \|g'(\theta_2 h_2) - t^2 f''(x) h_1\| \, |h_2| = \\ = \sup_{0<\theta_2<1} \|(f'(x + t(h_1 + \theta_2 h_2)) &- f'(x + t\theta_2 h_2))t - t^2 f''(x) h_1\| \, |h_2| . \end{aligned}$$

By definition of the derivative mapping we can write that

$$f'\big(x + t(h_1 + \theta_2 h_2)\big) = f'(x) + f''(x)\big(t(h_1 + \theta_2 h_2)\big) + o(t)$$

and

$$f'(x + t\theta_2 h_2) = f'(x) + f''(x)(t\theta_2 h_2) + o(t)$$

as $t \to 0$. Taking this relation into account, one can continue the preceding computation, finding after cancellation that

$$|F_t(h_1, h_2) - t^2 f''(x)(h_1, h_2)| = o(t^2)$$

as $t \to 0$. But this equality means that

$$f''(x)(h_2, h_2) = \lim_{t \to 0} \frac{F_t(h_1, h_2)}{t^2} .$$

Since it is obvious that $F_t(h_1, h_2) = F_t(h_2, h_1)$, it follows from this relation that $f''(x)(h_1, h_2) = f''(x)(h_2, h_1)$.

One can now complete the proof of the proposition by induction, repeating verbatim what was said in the proof that the values of the mixed partial derivatives are independent of the order of differentiation. □

Thus we have shown that the nth differential of the mapping (10.58) at the point $x \in U$ is a symmetric n-linear transformation

$$f^{(n)}(x) \in \mathcal{L}(TX_x, \dots, TX_x; TY_{f(x)}) \sim \mathcal{L}(X, \dots, X; Y)$$

whose value on the set $(h_1, \dots, h_n)$ of vectors $h_i \in TX_x = X$, $i = 1, \dots, n$, can be computed by formula (10.67).

If X is a finite-dimensional space having a basis $\{e_1, \dots, e_k\}$ and $h_j = h_j^i e_i$ is the expansion of the vector h_j, $j = 1, \dots, n$, with respect to that basis, then by the multilinearity of $f^{(n)}(x)$ we can write

$$f^{(n)}(x)(h_1, \dots, h_n) = f^{(n)}(x)(h_1^{i_1} e_{i_1}, \dots, h_n^{i_n} e_{i_n}) = \\ = f^{(n)}(x)(e_{i_1}, \dots, e_{i_n}) h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} .$$

Using our earlier notation $\partial_{i_1 \cdots i_n} f(x)$ for $D_{e_1} \cdots D_{e_n} f(x)$, we find finally that

$$f^{(n)}(x)(h_1, \dots, h_n) = \partial_{i_1 \cdots i_n} f(x) h_1^{i_1} \cdots h_n^{i_n} ,$$

where as usual summation extends over the repeated indices on the right-hand side within their range of variation, that is, from 1 to k.

Let us agree to use the following abbreviation:

$$f^{(n)}(x)(h, \dots, h) =: f^{(n)}(x) h^n . \qquad (10.68)$$

In particular, if we are discussing a finite-dimensional space X and $h = h^i e_i$, then

$$f^{(n)}(x) h^n = \partial_{i_1 \cdots i_n} f(x) h^{i_1} \cdot \ldots \cdot h^{i_n} ,$$

which is already very familiar to us from the theory of numerical functions of several variables.

10.5.4 Some Remarks

In connection with the notation (10.68) consider the following example, which is quite useful and will be used in the next section.

Example. Let $A \in \mathcal{L}(X_1, \dots, X_n; Y)$, that is, $y = A(x_1, \dots, x_n)$ is a continuous n-linear transformation from the product of the normed vector spaces $X_1, \dots, X_n$ into the normed vector space Y.

It was shown in Example 5 of Sect. 10.4 that A is a differentiable mapping $A : X_1 \times \cdots \times X_n \to Y$ and

$$A'(x_1,\dots,x_n)(h_1,\dots,h_n) = \\ = A(h_1,x_2,\dots,x_n) + \cdots + A(x_1,\dots,x_{n-1},h_n)\,.$$

Thus, if $X_1 = \cdots = X_n = X$ and A is symmetric, then

$$A'(x,\dots,x)(h,\dots,h) = nA(\underbrace{x,\dots,x}_{n-1},h) =: (nAx^{n-1})h\,.$$

Hence, if we consider the function $F : X \to Y$ defined by the condition

$$X \ni x \mapsto F(x) = A(x,\dots,x) =: Ax^n\,,$$

it turns out to be differentiable and

$$F'(x)h = (nAx^{n-1})h\,,$$

that is, in this case

$$F'(x) = nAx^{n-1}\,,$$

where $Ax^{n-1} := A(\underbrace{x,\dots,x}_{n-1},\cdot)$.

In particular, if the mapping (10.58) has a differential $f^{(n)}(x)$ at a point $x \in U$, then the function $F(h) = f^{(n)}(x)h^n$ is differentiable, and

$$F'(h) = nf^{(n)}(x)h^{n-1}\,. \tag{10.69}$$

To conclude our discussion of the concept of an nth-order derivative, it is useful to add the remark that if the original function (10.58) is defined on a set U in a space X that is the direct product of normed spaces $X_1,\dots,X_m$, one can speak of the first-order partial derivatives $\partial_1 f(x),\dots,\partial_m f(x)$ of f with respect to the variables $x_i \in X_i$, $i = 1,\dots,m$, and the higher-order partial derivatives $\partial_{i_1\cdots i_n} f(x)$.

On the basis of Theorem 2 of Sect. 10.4, we obtain by induction in this case that *if all the partial derivatives $\partial_{i_1\cdots i_n} f(x)$ of a mapping $f : U \to Y$ are continuous at a point $x \in X = X_1 \times \cdots \times X_m$, then the mapping f has an n-th order differential $f^{(n)}(x)$ at that point.*

If we also take account of the result of Example 2 from the same section, we can conclude that *the mapping $U \ni x \mapsto f^{(n)}(x) \in \mathcal{L}(\underbrace{X,\dots,X}_{n \text{ factors}};Y)$ is continuous if and only if all the nth-order partial derivatives $U \ni x \mapsto \partial_{i_1\cdots i_n} f(x) \in \mathcal{L}(X_{i_1},\dots,X_{i_n};Y)$ of the original mapping $f : U \to Y$ are continuous* (or, what is the same, the partial derivatives of all orders up to n inclusive are continuous).

The class of mappings (10.58) having continuous derivatives up to order n inclusive in U is denoted $C^{(n)}(U,Y)$, or, where no confusion can arise, by the briefer symbol $C^{(n)}(U)$ or even $C^{(n)}$.

In particular, if $X = X_1 \times \cdots \times X_n$, the conclusion reached above can be written in abbreviated form as

$$(f \in C^{(n)}) \iff (\partial_{i_1 \cdots i_n} f \in C,\ i_1, \dots, i_n = 1, \dots, m)\ ,$$

where C, as always, denotes the corresponding set of continuous functions.

10.5.5 Problems and Exercises

1. Carry out the proof of Eq. (10.64) in full.

2. Give the details at the end of the proof that $f^{(n)}(x)$ is symmetric.

3. a) Show that if the functions $D_{h_1}D_{h_2}f$ and $D_{h_2}D_{h_1}f$ are defined and continuous at a point $x \in U$ for a pair of vectors h_1, h_2 and the mapping (10.58) in the domain U, then the equality $D_{h_1}D_{h_2}f(x) = D_{h_2}D_{h_1}f(x)$ holds.

b) Show using the example of a numerical function $f(x, y)$ that, although the continuity of the mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ implies by a) that they are equal at this point, it does not in general imply that the second differential of the function exists at the point.

c) Show that, although the existence of $f^{(2)}(x, y)$, guarantees that the mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are equal, it does not in general guarantee that they are continuous at that point.

4. Let $A \in \mathcal{L}(X, \dots, X; Y)$ where A is a symmetric n-linear transformation. Find the successive derivatives of the function $x \mapsto Ax^n := A(x, \dots, x)$ up to order $n+1$ inclusive.

10.6 Taylor's Formula and the Study of Extrema

10.6.1 Taylor's Formula for Mappings

Theorem 1. *If a mapping $f : U \to Y$ from a neighborhood $U = U(x)$ of a point x in a normed space X into a normed space Y has derivatives up to order $n - 1$ inclusive in U and has an n-th order derivative $f^{(n)}(x)$ at the point x, then*

$$f(x+h) = f(x) + f'(x)h + \cdots + \frac{1}{n!}f^{(n)}(x)h^n + o(|h|^n) \qquad (10.70)$$

as $h \to 0$.

Equality (10.70) is one of the varieties of Taylor's formula, written here for rather general classes of mappings.

Proof. We prove Taylor's formula by induction.

For $n = 1$ it is true by definition of $f'(x)$.

Assume formula (10.70) is true for some $(n-1) \in \mathbb{N}$.

Then by the mean-value theorem, formula (10.69) of Sect. 10.5, and the induction hypothesis, we obtain

$$\Big|f(x+h) - \big(f(x) + f'(x)h + \cdots + \frac{1}{n!}f^{(n)}(x)h^n\big)\Big| \le$$
$$\le \sup_{0<\theta<1} \Big\|f'(x+\theta h) - \Big(f'(x) + f''(x)(\theta h) + \cdots$$
$$\cdots + \frac{1}{(n-1)!}f^{(n)}(x)(\theta h)^{n-1}\Big)\Big\|\,|h| = o\big(|\theta h|^{n-1}\big)|h| = o\big(|h|^n\big)$$

as $h \to 0$. □

We shall not take the time here to discuss other versions of Taylor's formula, which are sometimes quite useful. They were discussed earlier in detail for numerical functions. At this point we leave it to the reader to derive them (see, for example, Problem 1 below).

10.6.2 Methods of Studying Interior Extrema

Using Taylor's formula, we shall exhibit necessary conditions and also sufficient conditions for an interior local extremum of real-valued functions defined on an open subset of a normed space. As we shall see, these conditions are analogous to the differential conditions already known to us for an extremum of a real-valued function of a real variable.

Theorem 2. *Let $f : U \to \mathbb{R}$ be a real-valued function defined on an open set U in a normed space X and having continuous derivatives up to order $k-1 \ge 1$ inclusive in a neighborhood of a point $x \in U$ and a derivative $f^{(k)}(x)$ of order k at the point x itself.*

If $f'(x) = 0, \dots, f^{(k-1)}(x) = 0$ and $f^{(k)}(x) \neq 0$, then for x to be an extremum of the function f it is:

necessary that k be even and that the form $f^{(k)}(x)h^k$ be semidefinite,[4] *and*

sufficient that the values of the form $f^{(k)}(x)h^k$ on the unit sphere $|h| = 1$ be bounded away from zero; moreover, x is a local minimum if the inequalities

$$f^{(k)}(x)h^k \ge \delta > 0$$

hold on that sphere, and a local maximum if

$$f^{(k)}(x)h^k \le \delta < 0\,.$$

[4] This means that the form $f^{(k)}(x)h^k$ cannot take on values of opposite signs, although it may vanish for some values $h \neq 0$. The equality $f^{(i)}(x) = 0$, as usual, is understood to mean that $f^{(i)}(x)h = 0$ for every vector h.

Proof. For the proof we consider the Taylor expansion (10.70) of f in a neighborhood of x. The assumptions enable us to write

$$f(x+h) - f(x) = \frac{1}{k!} f^{(k)}(x)h^k + \alpha(h)|h|^k ,$$

where $\alpha(h)$ is a real-valued function, and $\alpha(h) \to 0$ as $h \to 0$.

We first prove the necessary conditions.

Since $f^{(k)}(x) \neq 0$, there exists a vector $h_0 \neq 0$ on which $f^{(k)}(x)h_0^k \neq 0$. Then for values of the real parameter t sufficiently close to zero,

$$f(x+th_0) - f(x) = \frac{1}{k!} f^{(k)}(x)(th_0)^k + \alpha(th_0)|th_0|^k =$$
$$= \Big(\frac{1}{k!} f^{(k)}(x)h_0^k \pm \alpha(th_0)|h_0|^k\Big)t^k$$

and the expression in the outer parentheses has the same sign as $f^{(k)}(x)h_0^k$.

For x to be an extremum it is necessary for the left-hand side (and hence also the right-hand side) of this last equality to be of constant sign when t changes sign. But this is possible only if k is even.

This reasoning shows that if x is an extremum, then the sign of the difference $f(x+th_0) - f(x)$ is the same as that of $f^{(k)}(x)h_0^k$ for sufficiently small t; hence in that case there cannot be two vectors h_0, h_1 at which the form $f^{(k)}(x)$ assumes values with opposite signs.

We now turn to the proof of the sufficiency conditions. For definiteness we consider the case when $f^{(k)}(x)h^k \geq \delta > 0$ for $|h| = 1$. Then

$$f(x+h) - f(x) = \frac{1}{k!} f^{(k)}(x)h^k + \alpha(h)|h|^k =$$
$$= \Big(\frac{1}{k!} f^{(k)}(x)\Big(\frac{h}{|h|}\Big)^k + \alpha(h)\Big)|h|^k \geq \Big(\frac{1}{k!}\delta + \alpha(h)\Big)|h|^k ,$$

and, since $\alpha(h) \to 0$ as $h \to 0$, the last term in this inequality is positive for all vectors $h \neq 0$ sufficiently close to zero. Thus, for all such vectors h,

$$f(x+h) - f(x) > 0 ,$$

that is, x is a strict local minimum.

The sufficient condition for a strict local maximum is verified similiarly. □

Remark 1. If the space X is finite-dimensional, the unit sphere $S(x,1)$ with center at $x \in X$, being a closed bounded subset of X, is compact. Then the continuous function $f^{(k)}(x)h^k = \partial_{i_1 \dots i_k} f(x)h^{i_1} \cdot \ldots \cdot h^{i_k}$ (a k-form) has both a maximal and a minimal value on $S(x,1)$. If these values are of opposite sign, then f does not have an extremum at x. If they are both of the same sign, then, as was shown in Theorem 2, there is an extremum. In the latter case, a sufficient condition for an extremum can obviously be stated as the equivalent requirement that the form $f^{(k)}(x)h^k$ be either positive- or negative-definite.

It was this form of the condition that we encountered in studying real-valued functions on $\mathbb{R}^n$.

Remark 2. As we have seen in the example of functions $f : \mathbb{R}^n \to \mathbb{R}$, the semi-definiteness of the form $f^{(k)}h^k$ exhibited in the necessary conditions for an extremum is not a sufficient criterion for an extremum.

Remark 3. In practice, when studying extrema of differentiable functions one normally uses only the first or second differentials. If the uniqueness and type of extremum are obvious from the meaning of the problem being studied, one can restrict attention to the first differential when seeking an extremum, simply finding the point x where $f'(x) = 0$.

10.6.3 Some Examples

Example 1. Let $L \in C^{(1)}(\mathbb{R}^3, \mathbb{R})$ and $f \in C^{(1)}([a,b], \mathbb{R})$. In other words, $(u^1, u^2, u^3) \mapsto L(u^1, u^2, u^3)$ is a continuously differentiable real-valued function defined in $\mathbb{R}^3$ and $x \mapsto f(x)$ a smooth real-valued function defined on the closed interval $[a,b] \subset \mathbb{R}$.

Consider the function

$$F : C^{(1)}([a,b], \mathbb{R}) \to \mathbb{R} \tag{10.71}$$

defined by the relation

$$C^{(1)}([a,b], \mathbb{R}) \ni f \mapsto F(f) = \int_a^b L(x, f(x), f'(x))\, dx \in \mathbb{R}\,. \tag{10.72}$$

Thus, (10.71) is a real-valued functional defined on the set of functions $f \in C(1)([a,b], \mathbb{R})$.

The basic variational principles connected with motion are known in physics and mechanics. According to these principles, the actual motions are distinguished among all the conceivable motions in that they proceed along trajectories along which certain functionals have an extremum. Questions connected with the extrema of functionals are central in optimal control theory. Thus, finding and studying the extrema of functionals is a problem of intrinsic importance, and the theory associated with it is the subject of a large area of analysis – the calculus of variations. We have already done a few things to make the transition from the analysis of the extrema of numerical functions to the problem of finding and studying extrema of functionals seem natural to the reader. However, we shall not go deeply into the special problems of variational calculus, but rather use the example of the functional

(10.72) to illustrate only the general ideas of differentiation and study of local extrema considered above.

We shall show that the functional (10.72) is a differentiable mapping and find its differential.

We remark that the function (10.72) can be regarded as the composition of the mappings

$$F_1 : C^{(1)}\big([a,b],\mathbb{R}\big) \to C\big([a,b],\mathbb{R}\big) \tag{10.73}$$

defined by the formula

$$F_1(f)(x) = L\big(x, f(x)f'(x)\big) \tag{10.74}$$

followed by the mapping

$$C\big([a,b],\mathbb{R}\big) \ni g \mapsto F_2(g) = \int_a^b g(x)\,\mathrm{d}x \in \mathbb{R}\,. \tag{10.75}$$

By properties of the integral, the mapping F_2 is obviously linear and continuous, so that its differentiability is clear.

We shall show that the mapping F_1 is also differentiable, and that

$$F_1'(f)h(x) = \partial_2 L\big(x, f(x), f'(x)\big)h(x) + \partial_3 L\big(x, f(x)f'(x)\big)h'(x) \tag{10.76}$$

for $h \in C^{(1)}\big([a,b],\mathbb{R}\big)$.

Indeed, by the corollary to the mean-value theorem, we can write in the present case

$$\begin{aligned}\Big|L(u^1 &+ \Delta^1, u^2+\Delta^2, u^3+\Delta^3) - L(u^1,u^2,u^3) - \\ &- \sum_{i=1}^{3} \partial_i L(u^1,u^2,u^3)\Delta^i\Big| \le \sup_{0<\theta<1} \|(\partial_1 L(u+\theta\Delta) - \partial_1 L(u)\ , \\ &\partial_2\ L(u+\theta\Delta) - \partial_2 L(u),\ \partial_3 L(u+\theta\Delta) - \partial_3 L(u))\| \cdot |\Delta| \le \\ &\qquad \le 3 \max_{\substack{0\le\theta\le 1\\ i=1,2,3}} |\partial_i L(u+\theta u) - \partial_i L(u)| \cdot \max_{i=1,2,3} |\Delta^i|\ ,\end{aligned} \tag{10.77}$$

where $u = (u^1,u^2,u^3)$ and $\Delta = (\Delta^1,\Delta^2,\Delta^3)$.

If we now recall that the norm $|f|_{C^{(1)}}$ of the function f in $C^{(1)}\big([a,b],\mathbb{R}\big)$ is $\max\big\{|f|_C, |f'|_C\big\}$ (where $|f|_C$ is the maximum absolute value of the function on the closed interval $[a,b]$), then, setting $u^1 = x$, $u^2 = f(x)$, $u^3 = f'(x)$, $\Delta^1 = 0$, $\Delta^2 = h(x)$, and $\Delta^3 = h'(x)$, we obtain from inequality (10.77), taking account of the uniform continuity of the functions $\partial_i L(u^1,u^2,u^3)$, $i = 1,2,3$, on bounded subsets of $\mathbb{R}^3$, that

$$\begin{aligned}\max_{a\le x\le b} &|L(x, f(x)+h(x), f'(x)+h'(x)) - L(x, f(x), f'(x)) - \\ &- \partial_2 L(x, f(x), f'(x))h(x) - \partial_3 L(X, f(x)f'(x))h'(x)| = \\ &\qquad = o\big(|h|_{C^{(1)}}\big) \text{ as } |h|_{C^{(1)}} \to 0\,.\end{aligned}$$

But this means that Eq. (10.76) holds.

By the chain rule for differentiating a composite function, we now conclude that the functional (10.72) is indeed differentiable, and

$$F'(f)h = \int_a^b \big(\big(\partial_2 L(x, f(x) f'(x)\big) h(x) + \partial_3 L\big(x, f(x), f'(x)\big)\big) h'(x)\big) \, \mathrm{d}x \,. \tag{10.78}$$

We often consider the restriction of the functional (10.72) to the affine space consisting of the functions $f \in C^{(1)}\big([a,b],\mathbb{R}\big)$ that assume fixed values $f(a) = A$, $f(b) = B$ at the endpoints of the closed interval $[a,b]$. In this case, the functions h in the tangent space $TC_f^{(1)}$ must have the value zero at the endpoints of the closed interval $[a,b]$. Taking this fact into account, we may integrate by parts in (10.78) and bring it into the form

$$F'(f)h = \int_a^b \Big(\partial_2 L\big(x, f(x), f'(x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\partial_3 L\big(x, f(x) f'(x)\big)\Big) h(x) \, \mathrm{d}x \,, \tag{10.79}$$

of course under the assumption that L and f belong to the corresponding class $C^{(2)}$.

In particular, if f is an extremum (extremal) of such a functional, then by Theorem 2 we have $F'(f)h = 0$ for every function $h \in C^{(1)}\big([a,b],\mathbb{R}\big)$ such that $h(a) = h(b) = 0$. From this and relation (10.79) one can easily conclude (see Problem 3 below) that the function f must satisfy the equation

$$\partial_2 L(x, f(x), f'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\partial_3 L(x, f(x), f'(x)) = 0 \,. \tag{10.80}$$

This is a frequently-encountered form of the equation known in the calculus of variations as the *Euler–Lagrange equation.*

Let us now consider some specific examples.

Example 2. The shortest-path problem.

Among all the curves in a plane joining two fixed points, find the curve that has minimal length.

The answer in this case is obvious, and it rather serves as a check on the formal computations we will be doing later.

We shall assume that a fixed Cartesian coordinate system has been chosen in the plane, in which the two points are, for example, $(0,0)$ and $(1,0)$. We confine ourselves to just the curves that are the graphs of functions $f \in C^{(1)}\big([0,1],\mathbb{R}\big)$ assuming the value zero at both ends of the closed interval $[0,1]$. The length of such a curve

$$F(f) = \int_0^1 \sqrt{1 + (f')^2(x)} \, \mathrm{d}x \tag{10.81}$$

depends on the function f and is a functional of the type considered in Example 1. In this case the function L has the form

$$L(u^1, u^2, u^3) = \sqrt{1 + (u^3)^2}\ ,$$

and therefore the necessary condition (10.80) for an extremal here reduces to the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f'(x)}{\sqrt{1 + (f')^2(x)}}\right) = 0\ ,$$

from which it follows that

$$\frac{f'(x)}{\sqrt{1 + (f')^2(x)}} \equiv \text{const} \tag{10.82}$$

on the closed interval $[0, 1]$.

Since the function $\frac{u}{\sqrt{1+u^2}}$ is not constant on any interval, Eq. (10.82) is possible only if $f'(x) \equiv$ const on $[a, b]$. Thus a smooth extremal of this problem must be a linear function whose graph passes through the points $(0, 0)$ and $(1, 0)$. It follows that $f(x) \equiv 0$, and we arrive at the closed interval of the line joining the two given points.

Example 3. The brachistochrone problem.

The classical brachistochrone problem, posed by Johann Bernoulli I in 1696, was to find the shape of a track along which a point mass would pass from a prescribed point P_0 to another fixed point P_1 at a lower level under the action of gravity in the shortest time.

We neglect friction, of course. In addition, we shall assume that the trivial case in which both points lie on the same vertical line is excluded.

In the vertical plane passing through the points P_0 and P_1 we introduce a rectangular coordinate system such that P_0 is at the origin, the x-axis is directed vertically downward, and the point P_1 has positive coordinates (x_1, y_1). We shall find the shape of the track among the graphs of smooth functions defined on the closed interval $[0, x_1]$ and satisfying the condition $f(0) = 0$, $f(x_1) = y_1$. At the moment we shall not take time to discuss this by no means uncontroversial assumption (see Problem 4 below).

If the particle began its descent from the point P_0 with zero velocity, the law of variation of its velocity in these coordinates can be written as

$$v = \sqrt{2gx} \tag{10.83}$$

Recalling that the differential of the arc length is computed by the formula

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + (f')^2(x)}\,\mathrm{d}x\ , \tag{10.84}$$

we find the time of descent

$$F(f) = \frac{1}{\sqrt{2g}} \int_0^{x_1} \sqrt{\frac{1 + (f')^2(x)}{x}}\, \mathrm{d}x \tag{10.85}$$

along the trajectory defined by the graph of the function $y = f(x)$ on the closed interval $[0, x_1]$.

For the functional (10.85)

$$L(u^1, u^2, u^3) = \sqrt{\frac{1 + (u^3)^2}{u^1}}\,,$$

and therefore the condition (10.80) for an extremum reduces in this case to the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f'(x)}{\sqrt{x(1 + (f')^2(x))}}\right) = 0\,,$$

from which it follows that

$$\frac{f'(x)}{\sqrt{1 + (f')^2(x)}} = c\sqrt{x}\,, \tag{10.86}$$

where c is a nonzero constant, since the points are not both on the same vertical line.

Taking account of (10.84), we can rewrite (10.86) in the form

$$\frac{\mathrm{d}y}{\mathrm{d}s} = c\sqrt{x}\,. \tag{10.87}$$

However, from the geometric point of view

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\varphi\,, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\varphi\,, \tag{10.88}$$

where φ is the angle between the tangent to the trajectory and the positive x-axis.

By comparing Eq. (10.87) with the second equation in (10.88), we find

$$x = \frac{1}{c^2}\sin^2\varphi\,. \tag{10.89}$$

But it follows from (10.88) and (10.89) that

$$\frac{\mathrm{d}y}{\mathrm{d}\varphi} = \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}\varphi} = \tan\varphi\frac{\mathrm{d}x}{\mathrm{d}\varphi} = \tan\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{\sin^2\varphi}{c^2}\right) = 2\frac{\sin^2\varphi}{c^2}\,,$$

from which we find

$$y = \frac{2}{c^2}(2\varphi - \sin 2\varphi) + b\,. \tag{10.90}$$

Setting $2/c^2 =: a$ and $2\varphi =: t$, we write relations (10.89) and (10.90) as

$$\begin{aligned} x &= a(1-\cos t)\,, \\ y &= a(t-\sin t)+b\,. \end{aligned} \tag{10.91}$$

Since $a \neq 0$, it follows that $x = 0$ only for $t = 2k\pi$, $k \in \mathbb{Z}$. It follows from the form of the function (10.91) that we may assume without loss of generality that the parameter value $t = 0$ corresponds to the point $P_0 = (0,0)$. In this case Eq. (10.90) implies $b = 0$, and we arrive at the simpler form

$$\begin{aligned} x &= a(1-\cos t)\,, \\ y &= a(t-\sin t) \end{aligned} \tag{10.92}$$

for the parametric definition of this curve.

Thus the brachistochrone is a cycloid having a cusp at the initial point P_0 where the tangent is vertical. The constant a, which is a scaling coefficient, must be chosen so that the curve (10.92) also passes through the point P_1. Such a choice, as one can see by sketching the curve (10.92), is by no means always unique, and this shows that the necessary condition (10.80) for an extremum is in general not sufficient. However, from physical considerations it is clear which of the possible values of the parameter a should be preferred (and this, of course, can be confirmed by direct computation).

10.6.4 Problems and Exercises

1. Let $f : U \to Y$ be a mapping of class $C^{(n)}(U;Y)$ from an open set U in a normed space X into a normed space Y. Suppose the closed interval $[x, x+h]$ is entirely contained in U, that f has a differential of order $(n+1)$ at the points of the open interval $]x, x+h[$, and that $\|f^{(n+1)}(\xi)\| \leq M$ at every point $\xi \in]x, x+h[$.

a) Show that the function

$$g(t) = f(x+th) - \Big(f(x) + f'(x)(th) + \cdots + \frac{1}{n!}f^{(n)}(x)(th)^n\Big)$$

is defined on the closed interval $[0,1] \subset \mathbb{R}$ and differentiable on the open interval $]0,1[$, and that the estimate

$$\|g'(t)\| \leq \frac{1}{n!}M|th|^n|h|$$

holds for every $t \in]0,1[$.

b) Show that $|g(1) - g(0)| \leq \frac{1}{(n+1)!}M|h|^{n+1}$.

c) Prove the following version of Taylor's formula:

$$\left|f(x+h) - \Big(f(x) + f'(x)h + \cdots + \frac{1}{n!}f^{(n)}(x)h^n\Big)\right| \leq \frac{M}{(n+1)!}|h|^{n+1}\,.$$

d) What can be said about the mapping $f : U \to Y$ if it is known that $f^{(n+1)}(x) \equiv 0$ in U?

2. a) If a symmetric n-linear operator A is such that $Ax^n = 0$ for every vector $x \in X$, then $A(x_1, \dots, x_n) \equiv 0$, that is, A equals zero on every set $x_1, \dots, x_n$ of vectors in X.

b) If a mapping $f : U \to Y$ has an nth-order differential $f^{(n)}(x)$ at a point $x \in U$ and satisfies the condition

$$f(x+h) = L_0 + L_1 h + \cdots + \frac{1}{n!} L_n h^n + \alpha(h)|h|^n ,$$

where L_i, $i = 0, 1, \dots, n$, are i-linear operators, and $\alpha(h) \to 0$ as $h \to 0$, then $L_i = f^{(i)}(x)$, $i = 0, 1, \dots, n$.

c) Show that the existence of the expansion for f given in the preceding problem does not in general imply the existence of the n-th order differential $f^{(n)}(x)$ (for $n > 1$) for the function at the point x.

d) Prove that the mapping $\mathcal{L}(X;Y) \ni A \mapsto A^{-1} \in \mathcal{L}(X;Y)$ is infinitely differentiable in its domain of definition, and that $(A^{-1})^{(n)}(A)(h_1, \dots, h_n) = (-1)^n A^{-1} h_1 A^{-1} h_2 \cdot \ldots \cdot A^{-1} h_n A^{-1}$.

3. a) Let $\varphi \in C\Big([a,b], \mathbb{R}\Big)$. Show that if the condition

$$\int_a^b \varphi(x) h(x)\,\mathrm{d}x = 0$$

holds for every function $h \in C^{(2)}\Big([a,b], \mathbb{R}\Big)$ such that $h(a) = h(b) = 0$, then $\varphi(x) \equiv 0$ on $[a,b]$.

b) Derive the Euler–Lagrange equation (10.80) as a necessary condition for an extremum of the functional (10.72) restricted to the set of functions $f \in C^{(2)}\Big([a,b], \mathbb{R}\Big)$ assuming prescribed values at the endpoints of the closed interval $[a,b]$.

4. Find the shape $y = f(x)$, $a \le x \le b$, of a meridian of the surface of revolution (about the x-axis) having minimal area among all surfaces of revolution having circles of prescribed radius r_a and r_b as their sections by the planes $x = a$ and $x = b$ respectively.

5. a) The function L in the brachistochrone problem does not satisfy the conditions of Example 1, so that we cannot justify a direct application of the results of Example 1 in this case. Show by repeating the derivation of formula (10.79) with necessary modifications that this equation and Eq. (10.80) remain valid in this case.

b) Does the equation of the brachistochrone change if the particle starts from the point P_0 with a nonzero initial velocity (the motion is frictionless in a closed pipe)?

c) Show that if P is an arbitrary point of the brachistochrone corresponding to the pair of points P_0, P_1, the arc of that brachistochrone from P_0 to P is the brachistochrone of the pair P_0, P.

d) The assumption that the brachistochrone corresponding to a pair of points P_0, P_1 can be written as $y = f(x)$, is not always justified, as was revealed by the final formulas (10.92). Show by using the result of c) that the derivation of (10.92) can be carried out without any such assumption as to the global structure of the brachistochrone.

e) Locate a point P_1 such that the brachistochrone corresponding to the pair of points P_0, P_1 in the coordinate system introduced in Example 3 cannot be written in the form $y = f(x)$.

f) Locate a point P_1 such that the brachistochrone corresponding to the pair of points P_0, P_1 in the coordinate system introduced in Example 3) has the form $y = f(x)$, and $f \notin C^{(1)}\Big([a,b],\mathbb{R}\Big)$. Thus it turns out that in this case the functional (10.85) we are interested in has a greatest lower bound on the set $C^{(1)}\Big([a,b],\mathbb{R}\Big)$, but not a minimum.

g) Show that the brachistochrone of a pair of points P_0, P_1 of space is a smooth curve.

6. Let us measure the distance $d(P_0, P_1)$ of the point P_0 of space from the point P_1 in a homogeneous gravitational field by the time required for a point mass to move from one point to the other along the brachistochrone corresponding to the points.

a) Find the distance from the point P_0 to a fixed vertical line, measured in this sense.

b) Find the asymptotic behavior of the function $d(P_0, P_1)$ as the point P_1 is raised along a vertical line, approaching the height of the point P_0.

c) Determine whether the function $d(P_0, P_1)$ is a metric.

10.7 The General Implicit Function Theorem

In this concluding section of the chapter we shall illustrate practically all of the machinery we have developed by studying an implicitly defined function. The reader already has some idea of the content of the implicit theorem, its place in analysis, and its applications from Chap. 8. For that reason, we shall not go into detail here with preliminary explanations of the essence of the matter preceding the formalism. We note only that this time the implicitly defined function will be constructed by an entirely different method, one that relies on the contraction mapping principle. This method is often used in analysis and is quite useful because of its computational efficiency.

Theorem. *Let X, Y, and Z be normed spaces (for example, $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^k$), Y being a complete space. Let $W = \{(x,y) \in X \times Y \mid |x - x_0| < \alpha \wedge |y - y_0| < \beta\}$ be a neighborhood of the point (x_0, y_0) in the product $X \times Y$ of the spaces X and Y.*

Suppose that the mapping $F : W \to Z$ satisfies the following conditions:

1. $F(x_0, y_0) = 0$;
2. $F(x,y)$ *is continuous at* (x_0, y_0);

3. *$F'(x,y)$ is defined in W and continuous at (x_0, y_0);*
4. *$F'_y(x_0, y_0)$ is an invertible[5] transformation.*

Then there exists a neighborhood $U = U(x_0)$ of $x_0 \in X$, a neighborhood $V = V(y_0)$ of $y_0 \in Y$, and a mapping $f : U \to V$ such that:

1′. *$U \times V \subset W$;*
2′. *$\big(F(x,y) = 0$ in $U \times V\big) \Leftrightarrow \big(y = f(x)$, where $x \in U$ and $f(x) \in V\big)$;*
3′. *$y_0 = f(x_0)$;*
4′. *f is continuous at x_0.*

In essence, this theorem asserts that if the linear mapping F'_y is invertible at a point (hypothesis 4), then in a neighborhood of this point the relation $F(x,y) = 0$ is equivalent to the functional dependence $y = f(x)$ (conclusion 2′).

Proof. 1^0 To simplify the notation and obviously with no loss of generality, we may assume that $x_0 = 0$, $y_0 = 0$, and consequently

$$W = \{(x,y) \in X \times Y \,|\, |x| < \alpha \wedge |y| < \beta\} .$$

2^0 The main role in the proof is played by the auxiliary family of functions

$$g_x(y) := y - \big(F'_y(0,0)\big)^{-1} \cdot F(x,y) , \tag{10.93}$$

which depend on the parameter $x \in S$, $|x| < \alpha$, and are defined on the set $\{y \in Y \,|\, |y| < \beta\}$.

Let us discuss formula (10.93). We first determine whether the mappings g_x are unambiguously defined and where their values lie.

The mapping F is defined for $(x,y) \in W$, and its value $F(x,y)$ at the pair (x,y) lies in Z. The partial derivative $F'_y(x,y)$ at any point $(x,y) \in W$, as we know, is a continuous linear mapping from Y into Z.

By hypothesis 4 the mapping $F'_y(0,0) : Y \to Z$ has a continuous inverse $\big(F'_y(0,0)\big)^{-1} : Z \to Y$. Hence the composition $\big(F'_y(0,0)\big)^{-1} \cdot F(x,y)$ really is defined, and its values lie in Y.

Thus, for any x in the α-neighborhood $B_X(0,\alpha) := \{x \in X \,|\, |x| < \alpha\}$ of the point $0 \in X$, the function g_x is a mapping $g_x : B_Y(0,\beta) \to Y$ from the β-neighborhood $B_Y(0,\beta) := \{y \in Y \,|\, |y| < \beta\}$ of the point $0 \in Y$ into Y.

The connection of the mappings (10.93) with the problem of solving the equation $F(x,y) = 0$ for y obviously consists of the following: the point y_x is a fixed point of g_x if and only if $F(x, y_x) = 0$.

Let us state this important observation firmly:

$$g_x(y_x) = y_x \Longleftrightarrow F(x, y_x) = 0 . \tag{10.94}$$

Thus, finding and studying the implicitly defined function $y = y_x = f(x)$ reduces to finding the fixed points of the mappings (10.93) and studying the way in which they depend on the parameter x.

[5] That is, $\exists\, [F'_y(x_0, y_0)]^{-1} \in \mathcal{L}(Z; Y)$.

3^0 We shall show that there exists a positive number $\gamma < \min\{\alpha, \beta\}$ such that for each $x \in X$ satisfying the condition $|x| < \gamma < \alpha$, the mapping $g_x : B_Y(0,\gamma) \to Y$ of the ball $B_Y(0,\gamma) := \{y \in Y \,|\, |y| < \gamma < \beta\}$ into Y is a contraction with a coefficient of contraction that does not exceed, say 1/2. Indeed, for each fixed $x \in B_X(0,\alpha)$ the mapping $g_x : B_Y(0,\beta) \to Y$ is differentiable, as follows from hypothesis 3 and the theorem on differentiation of a composite mapping. Moreover,

$$g'_x(y) = e_Y - \left(F'_y(0,0)\right)^{-1} \cdot \left(F'_y(x,y)\right) = \\ = \left(F'_y(0,0)\right)^{-1}\left(F'_y(0,0) - F'_y(x,y)\right) . \quad (10.95)$$

By the continuity of $F'_y(x,y)$ at the point $(0,0)$ (hypothesis 3), there exists a neighborhood $\{(x,y) \in X \times Y \,|\, |x| < \gamma < \alpha \wedge |y| < \gamma < \beta\}$ of $(0,0) \in X \times Y$ in which

$$\|g'_x(y)\| \le \|(F'_y(0,0))^{-1}\| \cdot \|F'_y(0,0) - F'_y(x,y)\| < \frac{1}{2} . \qquad (10.96)$$

Here we are using the relation

$$\left(F'_y(0,0)\right)^{-1} \in \mathcal{L}(Z;Y) , \text{ that is, } \|(F'_y(0,0))^{-1}\| < \infty .$$

Throughout the following we shall assume that $|x| < \gamma$ and $|y| < \gamma$, so that estimate (10.96) holds.

Thus, at any $x \in B_X(0,\gamma)$ and for any $y_1, y_2 \in B_Y(0,\gamma)$, by the mean-value theorem, we indeed now find that

$$|g_x(y_1) - g_x(y_2)| \le \sup_{\xi \in]y_1,y_2[} \|g'(\xi)\| \, |y_1 - y_2| < \frac{1}{2}|y_1 - y_2| . \qquad (10.97)$$

4^0. In order to assert the existence of a fixed point y_x for the mapping g_x, we need a complete metric space that maps into (but not necessarily onto) itself under this mapping.

We shall verify that for any ε satisfying $0 < \varepsilon < \gamma$ there exists $\delta = \delta(\varepsilon)$ in the open interval $]0,\gamma[$ such that for any $x \in B_X(0,\delta)$ the mapping g_x maps the closed ball $\overline{B}_y(0,\varepsilon)$ into itself, that is, $g_x\big(\overline{B}_Y(0,\varepsilon)\big) \subset \overline{B}_Y(0,\varepsilon)$.

Indeed, we first choose a number $\delta \in]0,\gamma[$ depending on ε such that

$$|g_x(0)| = |(F'_y(0,0))^{-1} \cdot F(x,0)\| \le \|(F'_y(0,0))^{-1}\| \, |F(x,0)| < \frac{1}{2}\varepsilon \quad (10.98)$$

for $|x| < \delta$.

This can be done by hypotheses 1 and 2, which guarantee that $F(0,0) = 0$ and $F(x,y)$ is continuous at $(0,0)$.

Now if $|x| < \delta(\varepsilon) < \gamma$ and $|y| \le \varepsilon < \gamma$, we find by (10.97) and (10.98) that

$$|g_x(y)| \le |g_x(y) - g_x(0)| + |g_x(0)| < \frac{1}{2}|y| + \frac{1}{2}\varepsilon < \varepsilon ,$$

and hence for $|x| < \delta(\varepsilon)$

$$g_x\big(\overline{B}_Y(0,\varepsilon)\big) \subset B_Y(0,\varepsilon) . \qquad (10.99)$$

Being a closed subset of the complete metric space Y, the closed ball $\overline{B}_Y(0,\varepsilon)$ is itself a complete metric space.

5^0 Comparing relations (10.97) and (10.99), we can now assert by the fixed-point principle (Sect. 9.7) that for each $x \in B_X(0,\delta(\varepsilon)) =: U$ there exists a unique point $y = y_x =: f(x) \in B_Y(0,\varepsilon) =: V$ that is a fixed point of the mapping $g_x : \overline{B}_Y(0,\varepsilon) \to \overline{B}_Y(0,\varepsilon)$.

By the basic relation (10.94), it follows from this that the function $f : U \to V$ so constructed has property $2'$ and hence also property $3'$, since $F(0,0) = 0$ by hypothesis 1.

Property $1'$ of the neighborhoods U and V follows from the fact that, by construction, $U \times V \subset B_X(0,\alpha) \times B_Y(0,\beta) = W$.

Finally, the continuity of the function $y = f(x)$ at $x = 0$, that is, property $4'$, follows from $2'$ and the fact that, as was shown in part 4^0 of the proof, for every $\varepsilon > 0$ $(\varepsilon < \gamma)$ there exists $\delta(\varepsilon) > 0$ $(\delta(\varepsilon) < \gamma)$ such that $g_x\big(\overline{B}_Y(0,\varepsilon)\big) \subset B_Y(0,\varepsilon)$ for any $x \in B_X(0,\delta(\varepsilon))$, that is, the unique fixed point $y_x = f(x)$ of the mapping $g_x : \overline{B}_Y(0,\varepsilon) \to \overline{B}_Y(0,\varepsilon)$ satisfies the condition $|f(x)| < \varepsilon$ for $|x| < \delta(\varepsilon)$. □

We have now proved the existence of the implicit function. We now prove a number of extensions of these properties of the function, generated by properties of the original function F.

Extension 1. (Continuity of the implicit function.) *If in addition to hypotheses 2 and 3 of the theorem it is known that the mappings $F : W \to Z$ and F'_y are continuous not only at the point (x_0, y_0) but in some neighborhood of this point, then the function $f : U \to V$ will be continuous not only at $x_0 \in U$ but in some neighborhood of this point.*

Proof. By properties of the mapping $\mathcal{L}(Y;Z) \ni A \mapsto A^{-1} \in \mathcal{L}(Z;Y)$ it follows from hypotheses 3 and 4 of the theorem (see Example 6 of Sect. 10.3) that at each point (x,y) in some neighborhood of (x_0,y_0) the transformation $f'_y(x,y) \in \mathcal{L}(Y;Z)$ is invertible. Thus under the additional hypothesis that F is continuous all points $(\tilde{x},\tilde{y})$ of the form $(x, f(x))$ in some neighborhood of (x_0,y_0) satisfy hypotheses 1–4, previously satisfied only by the point (x_0,y_0).

Repeating the construction of the implicit function in a neighborhood of these points $(\tilde{x},\tilde{y})$, we would obtain a function $y = \tilde{f}(x)$ that is continuous at $\tilde{x}$ and by $2'$ would coincide with the function $y = f(x)$ in some neighborhood of x. But that means that f itself is continuous at $\tilde{x}$. □

Extension 2. (Differentiability of the implicit function.) *If in addition to the hypotheses of the theorem it is known that a partial derivative $F'_x(x,y)$ exists in some neighborhood W of (x_0,y_0) and is continuous at (x_0,y_0), then the function $y = f(x)$ is differentiable at x_0, and*

$$f'(x_0) = -\big(F'_y(x_0,y_0)\big)^{-1} \cdot \big(F'_x(x_0,y_0)\big)\,. \tag{10.100}$$

Proof. We verify immediately that the linear transformation $L \in \mathcal{L}(X;Y)$ on the right-hand side of formula (10.100) is indeed the differential of the function $y = f(x)$ at x_0.

As before, to simplify the notation, we shall assume that $x_0 = 0$ and $y_0 = 0$, so that $f(0) = 0$.

We begin with a preliminary computation.

$$\begin{aligned}
|f(x) - f(0) - Lx| = |f(x) - Lx| = \\
&= \left|f(x) + \left(F'_y(0,0)\right)^{-1} \cdot \left(F'_x(0,0)\right)x\right| = \\
&= \left|\left(F'_y(0,0)\right)^{-1}\left(F'_x(0,0)x + F'_y(0,0)f(x)\right)\right| = \\
&= \left|\left(F'_y(0,0)\right)^{-1}\left(F(x,f(x)) - F(0,0) - F'_x(0,0)x - F'_y(0,0)f(x)\right)\right| \le \\
&\le \left\|\left(F'_y(0,0)\right)^{-1}\right\|\left|\left(F(x,f(x)) - F(0,0) - F'_x(0,0)x - F'_y(0,0)f(x)\right)\right| \le \\
&\le \left\|\left(F'_y(0,0)\right)^{-1}\right\| \cdot \alpha\left(x,f(x)\right)\left(|x| + |f(x)|\right) ,
\end{aligned}$$

where $\alpha(x,y) \to 0$ as $(x,y) \to (0,0)$.

These relations have been written taking account of the relation $F\left(x,f(x)\right) \equiv 0$ and the fact that the continuity of the partial derivatives F'_x and F'_y at $(0,0)$ guarantees the differentiability of the function $F(x,y)$ at that point.

For convenience in writing we set $a := \|L\|$ and $b := \left\|\left(F'_y(0,0)\right)^{-1}\right\|$.

Taking account of the relations

$$|f(x)| = |f(x) - Lx + Lx| \le |f(x) - Lx| + |Lx| \le |f(x) - Lx| + a|x| ,$$

we can extend the preliminary computation just done and obtain the relation

$$|f(x) - Lx| \le b\alpha(x,f(x))\left((a+1)|x| + |f(x) - Lx|\right) ,$$

or

$$|f(x) - Lx| \le \frac{(a+1)b}{1 - b\alpha(x,f(x))}\alpha(x,f(x))|x| .$$

Since f is continuous at $x = 0$ and $f(0) = 0$, we also have $f(x) \to 0$ as $x \to 0$, and therefore $\alpha(x,f(x)) \to 0$ as $x \to 0$.

It therefore follows from the last inequality that

$$|f(x) - f(0) - Lx| = |f(x) - Lx| = o\left(|x|\right) \text{ as } x \to 0. \ \square$$

Extension 3. (Continuous differentiability of the implicit function.) *If in addition to the hypotheses of the theorem it is known that the mapping F has continuous partial derivatives F'_x and F'_y in some neighborhood W of (x_0, y_0), then the function $y = f(x)$ is continuously differentiable in some neighborhood of x_0, and its derivative is given by the formula*

$$f'(x) = -\left(F'_y(x,f(x))\right)^{-1} \cdot \left(F'_x(x,f(x))\right) . \tag{10.101}$$

Proof. We already know from formula (10.100) that the derivative $f'(x)$ exists and can be expressed in the form (10.101) at an individual point x at which the transformation $F'_y(x, f(x))$ is invertible.

It remains to be verified that under the present hypotheses the function $f'(x)$ is continuous in some neighborhood of $x = x_0$.

The bilinear mapping $(A, B) \mapsto A \cdot B$ – the product of linear transformations A and B – is a continuous function.

The transformation $B = -F'_x(x, f(x))$ is a continuous function of x, being the composition of the continuous functions $x \mapsto (x, f(x)) \mapsto -F'_x(x, f(x))$.

The same can be said about the linear transformation $A^{-1} = F'_y(x, f(x))$.

It remains only to recall (see Example 6 of Sect. 10.3) that the mapping $A^{-1} \mapsto A$ is also continuous in its domain of definition.

Thus the function $f'(x)$ defined by formula (10.101) is continuous in some neighborhood of $x = x_0$, being the composition of continuous functions. □

We can now summarize and state the following general proposition.

Proposition. *If in addition to the hypotheses of the implicit function theorem it is known that the function F belongs to the class $C^{(k)}(W, Z)$, then the function $y = f(x)$ defined by the equation $F(x, y) = 0$ belongs to $C^{(k)}(U, Y)$ in some neighborhood U of x_0.*

Proof. The proposition has already been proved for $k = 0$ and $k = 1$. The general case can now be obtained by induction from formula (10.101) if we observe that the mapping $\mathcal{L}(Y; Z) \ni A \mapsto A^{-1} \in \mathcal{L}(Z; Y)$ is (infinitely) differentiable and that when Eq. (10.101) is differentiated, the right-hand side always contains a derivative of f one order less than the left-hand side. Thus, successive differentiation of Eq. (10.101) can be carried out a number times equal to the order of smoothness of the function F. □

In particular, if

$$f'(x)h_1 = -\big(F'_y(x, f(x))\big)^{-1} \cdot \big(F'_x(x, f(x))\big)h_1 ,$$

then

$$\begin{aligned}
f''(x)(h_1, h_2) &= -\mathrm{d}\big(F'_y(x, f(x))\big)^{-1} h_2 F'_x(x, f(x))h_1 - \\
&\qquad - \big(F'_y(x, f(x))\big)^{-1} \mathrm{d}\big(F'_x(x, f(x))h_1\big)h_2 = \\
&= \big(F'_y(x, f(x))\big)^{-1} \mathrm{d}F'_y(x, f(x))h_2 \big(F'_y(x, f(x))\big)^{-1} F'_x(x, f(x))h_1 - \\
&\qquad - \big(F'_y(x, f(x))\big)^{-1} \big((F''_{xx}(x, f(x)) + F''_{yy}(x, f(x))f'(x))h_1\big)h_2 = \\
&= \big(F'_y(x, f(x))\big)^{-1} \big((F''_{yx}(x, f(x)) + F''_{yy}(x, f(x))f'(x))\big)h_2\big) \times \\
&\qquad \times \big(F'_y(x, f(x))\big)^{-1} F'_x(x, f(x))h_1\big)\big(F'_y(x, f(x))\big)^{-1} \times \\
&\qquad \times \big((F''_{xx}(x, f(x)) + F''_{xy}(x, f(x))f'(x))h_1\big)h_2 .
\end{aligned}$$

In less detailed, but more readable notation, this means that

$$f''(x)(h_1,h_2) = (F'_y)^{-1}\big[\big((F''_{yx} + F''_{yy}f')h_2\big)(F'_y)^{-1}F'_x h_1 - \\ - \big((F''_{xx} + F''_{yy}f')h_1\big)h_2\big] . \quad (10.102)$$

In this way one could theoretically obtain an expression for the derivative of an implicit function to any order; however, as can be seen even from formula (10.102), these expressions are generally too cumbersome to be conveniently used. Let us now see how these results can be made specific in the important special case when $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, and $Z = \mathbb{R}^n$.

In this case the mapping $z = F(x,y)$ has the coordinate representation

$$\begin{aligned} z^1 &= F^1(x^1,\dots,x^m,y^1,\dots,y^n) , \\ &\dots\dots\dots\dots\dots\dots \\ z^n &= F^n(x^1,\dots,x^m,y^1,\dots,y^n) . \end{aligned} \quad (10.103)$$

The partial derivatives $F'_x \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ and $F'_y \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$ of the mapping are defined by the matrices

$$F'_x = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^m} \\ \dots & \dots & \dots \\ \frac{\partial F^n}{\partial x^1} & \cdots & \frac{\partial F^n}{\partial x^m} \end{pmatrix} , \qquad F'_y = \begin{pmatrix} \frac{\partial F^1}{\partial y^1} & \cdots & \frac{\partial F^1}{\partial y^n} \\ \dots & \dots & \dots \\ \frac{\partial F^n}{\partial y^1} & \cdots & \frac{\partial F^n}{\partial y^n} \end{pmatrix} ,$$

computed at the corresponding point (x,y).

As we know, the condition that F'_x and F'_y be continuous is equivalent to the continuity of all the entries of these matrices.

The invertibility of the linear transformation $F'_y(x_0,y_0) \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$ is equivalent to the nonsingularity of the matrix that defines this transformation.

Thus, in the present case the implicit function theorem asserts that if

1) $F^1(x_0^1,\dots,x_0^m,y_0^1,\dots,y_0^n) = 0$,

$\dots\dots\dots\dots\dots\dots$

$F^n(x_0^1,\dots,x_0^m,y_0^1,\dots,y_0^n) = 0$;

2) $F^i(x^1,\dots,x^m,y^1,\dots,y^n)$, $i = 1,\dots,n$, are continuous functions at the point $(x_0^1,\dots,x_0^m,y_0^1,\dots,y_0^n) \in \mathbb{R}^m \times \mathbb{R}^n$;

3) all the partial derivatives $\frac{\partial F^i}{\partial y^j}(x^1,\dots,x^m,y^1,\dots,y^n)$, $i = 1,\dots,n$, $j = 1,\dots,n$, are defined in a neighborhood of $(x_0^1,\dots,x_0^m,y_0^1,\dots,y_0^n)$ and are continuous at this point;

4) the determinant

$$\begin{vmatrix} \frac{\partial F^1}{\partial y^1} & \cdots & \frac{\partial F^1}{\partial y^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F^n}{\partial y^1} & \cdots & \frac{\partial F^n}{\partial y^n} \end{vmatrix}$$

of the matrix F'_y is nonzero at the point $(x_0^1, \dots, x_0^m, y_0^1, \dots, y_0^n)$;

then there exist a neighborhood U of $x_0 = (x_0^1, \dots, x_0^m) \in \mathbb{R}^m$, a neighborhood V of $y_0 = (y_0^1, \dots, y_0^n) \in \mathbb{R}^n$, and a mapping $f : U \to V$ having a coordinate representation

$$\begin{aligned} y^1 &= f^1(x^1, \dots, x^m), \\ &\cdots\cdots\cdots\cdots \\ y^n &= f^n(x^1, \dots, x^m), \end{aligned} \tag{10.104}$$

such that

1′) inside the neighborhood $U \times V$ of $(x_0^1, \dots, x_0^m, y_0^1, \dots, y_0^n) \in \mathbb{R}^m \times \mathbb{R}^n$ the system of equations

$$\begin{cases} F^1(x^1, \dots, x^m, y^1, \dots, y^n) = 0, \\ \cdots\cdots\cdots\cdots \\ F^n(x^1, \dots, x^m, y^1, \dots, y^n) = 0 \end{cases}$$

is equivalent to the functional relation $f : U \to V$ expressed by (10.104);

2′)

$$\begin{aligned} y_0^1 &= f^1(x_0^1, \dots, x_0^m), \\ &\cdots\cdots\cdots\cdots \\ y_0^n &= f^n(x_0^1, \dots, x_0^m); \end{aligned}$$

3′) the mapping (10.104) is continuous at $(x_0^1, \dots, x_0^m, y_0^1, \dots, y_0^n)$.

If in addition it is known that the mapping (10.103) belongs to the class $C^{(k)}$, then, as follows from the proposition above, the mapping (10.104) will also belong to $C^{(k)}$, of course within its own domain of definition.

In this case formula (10.101) can be made specific, becoming the matrix equality

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^m} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F^1}{\partial y^1} & \cdots & \frac{\partial F^1}{\partial y^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F^n}{\partial y^1} & \cdots & \frac{\partial F^n}{\partial y^n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^m} \\ \cdots & \cdots & \cdots \\ \frac{\partial F^n}{\partial x^1} & \cdots & \frac{\partial F^n}{\partial x^m} \end{pmatrix},$$

in which the left-hand side is computed at $(x^1, \dots, x^m)$ and the right-hand side at the corresponding point $(x^1, \dots, x^m, y^1, \dots, y^n)$, where $y^i = f^i(x^1, \dots, x^m)$, $i = 1, \dots, n$.

If $n = 1$, that is, when the equation

$$F(x^1, \dots, x^m, y) = 0$$

is being solved for y, the matrix F'_y consists of a single entry – the number $\frac{\partial F}{\partial y}(x^1, \dots, x^m, y)$. In this case $y = f(x^1, \dots, x^m)$, and

$$\left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^m}\right) = -\left(\frac{\partial F}{\partial y}\right)^{-1}\left(\frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^m}\right). \tag{10.105}$$

In this case formula (10.102) also simplifies slightly; more precisely, it can be written in the following more symmetric form:

$$f''(x)(h_1, h_2) = -\frac{(F''_{xx} + F''_{xy}f')h_1 F'_y h_2 - (F''_{yx} + F''_{yy}f')h_2 F'_x h_1}{(F'_y)^2}. \tag{10.106}$$

And if $n = 1$ and $m = 1$, then $y = f(x)$ is a real-valued function of one real argument, and formulas (10.105) and (10.106) simplify to the maximum extent, becoming the numerical equalities

$$f'(x) = -\frac{F'_x}{F'_y}(x, y),$$
$$f''(x) = -\frac{(F''_{xx} + F''_{xy}f')F'_y - (F''_{yx} + F''_{yy}f')F'_x}{(F'_y)^2}(x, y)$$

for the first two derivatives of the implicit function defined by the equation $F(x, y) = 0$.

10.7.1 Problems and Exercises

1. a) Assume that, along with the function $f : U \to Y$ given by the implicit function theorem, we have a function $\tilde{f} : \tilde{U} \to Y$ defined in some neighborhood $\tilde{U}$ of x_0 and satisfying $y_0 = \tilde{f}(x_0)$ and $F(x, \tilde{f}(x)) \equiv 0$ in $\tilde{U}$. Prove that if $\tilde{f}$ is continuous at x_0, then the functions f and $\tilde{f}$ are equal on some neighborhood of x_0.

b) Show that the assertion in a) is generally not true without the assumption that $\tilde{f}$ is continuous at x_0.

2. Analyze once again the proof of the implicit function theorem and the extensions to it, and show the following.

a) If $z = F(x, y)$ is a continuously differentiable complex-valued function of the complex variables x and y, then the implicit function $y = f(x)$ defined by the equation $F(x, y) = 0$ is differentiable with respect to the complex variable x.

b) Under the hypotheses of the theorem X is not required to be a normed space, and may be any topological space.

3. a) Determine whether the form $f''(x)(h_1, h_2)$ defined by relation (10.102) is symmetric.

b) Write the forms (10.101) and (10.102) for the case of numerical functions $F(x^1, x^2, y)$ and $F(x, y^1, y^2)$ in matrix form.

c) Show that if $\mathbb{R} \ni t \mapsto A(t) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ is family of nonsingular matrices $A(t)$ depending on the parameter t in an infinitely smooth manner, then

$$\frac{\mathrm{d}^2 A^{-1}}{\mathrm{d}t^2} = 2A^{-1}\left(\frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}\right)^2 - A^{-1}\frac{\mathrm{d}^2 A}{\mathrm{d}t^2}A^{-1}\ , \quad \text{where } A^{-1} = A^{-1}(t)$$

denotes the inverse of the matrix $A = A(t)$.

4. a) Show that Extension 1 to the theorem is an immediate corollary of the stability conditions for the fixed point of the family of contraction mappings studied in Sect. 9.7.

b) Let $\{A_t : X \to X\}$ be a family of contraction mappings of a complete normed space into itself depending on the parameter t, which ranges over a domain Ω in a normed space T. Show that if $A_t(x) = \varphi(t, x)$ is a function of class $C^{(n)}(\Omega \times X, X)$, then the fixed point $x(t)$ of the mapping A_t belongs to class $C^{(n)}(\Omega, X)$ as a function of t.

5. a) Using the implicit function theorem, prove the following *inverse function theorem*.

Let $g : G \to X$ be a mapping from a neighborhood G of a point y_0 in a complete normed space Y into a normed space X. If

1^0 the mapping $x = g(y)$ is differentiable in G,

2^0 $g'(y)$ is continuous at y_0,

3^0 $g'(y_0)$ is an invertible transformation,

then there exists a neighborhood $V \subset Y$ of y_0 and a neighborhood $U \subset X$ of x_0 such that $g : V \to U$ is bijective, and its inverse mapping $f : U \to V$ is continuous in U and differentiable at x_0; moreover,

$$f'(x_0) = \left(g'(y_0)\right)^{-1} .$$

b) Show that if it is known, in addition to the hypotheses given in a), that the mapping g belongs to the class $C^{(n)}(V, U)$, then the inverse mapping f belongs to $C^{(n)}(U, V)$.

c) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping for which the matrix $f'(x)$ is nonsingular at every point $x \in \mathbb{R}^n$ and satisfies the inequality $\|(f')^{-1}(x)\| > C > 0$ with a constant C that is independent of x. Show that f is a bijective mapping.

d) Using your experience in solving c), try to give an estimate for the radius of a spherical neighborhood $U = B(x_0, r)$ centered at x_0 in which the mapping $f : U \to V$ studied in the inverse function theorem is necessarily defined.

6. a) Show that if the linear mappings $A \in \mathcal{L}(X; Y)$ and $B \in \mathcal{L}(X; \mathbb{R})$ are such that $\ker A \subset \ker B$ (here ker, as usual, denotes the kernel of a transformation), then there exists a linear mapping $\lambda \in \mathcal{L}(Y; \mathbb{R})$, such that $B = \lambda \cdot A$.

b) Let X and Y be normed spaces and $f : X \to \mathbb{R}$ and $g : X \to Y$ smooth functions on X with values in $\mathbb{R}$ and Y respectively. Let S be the smooth surface defined in X by the equation $g(x) = y_0$. Show that if $x_0 \in S$ is an extremum of the function $f\big|_S$, then any vector h tangent to S at X_0 simultaneously satisfies two conditions: $f'(x_0)h = 0$ and $g'(x_0)h = 0$.

c) Prove that if $x_0 \in S$ is an extremum of the function $f\big|_S$ then $f'(x_0) = \lambda \cdot g'(x_0)$, where $\lambda \in \mathcal{L}(Y; \mathbb{R})$.

d) Show how the classical *Lagrange necessary condition for an extremum with constraint* of a function on a smooth surface in $\mathbb{R}^n$ follows from the preceding result.

7. As is known, the equation $z^n + c_1 z^{n-1} + \cdots + c_n = 0$ with complex coefficients has in general n distinct complex roots. Show that the roots of the equation are smooth functions of the coefficients, at least where all the roots are distinct.

11 Multiple Integrals

11.1 The Riemann Integral over an n-Dimensional Interval

11.1.1 Definition of the Integral

a. Intervals in $\mathbb{R}^n$ and their Measure

Definition 1. The set $I = \{x \in \mathbb{R}^n \mid a^i \leq x^i \leq b^i,\ i = 1, \dots, n\}$ is called an *interval* or a *coordinate paralellepiped* in $\mathbb{R}^n$.

If we wish to note that the interval is determined by the points $a = (a^1, \dots, a^n)$ and $b = (b^1, \dots, b^n)$, we often denote it $I_{a,b}$, or, by analogy with the one-dimensional case, we write it as $a \leq x \leq b$.

Definition 2. To the interval $I = \{x \in \mathbb{R}^n \mid a^i \leq x^i \leq b^i,\ i = 1, \dots, n\}$ we assign the number $|I| := \prod\limits_{i=1}^{n} (b^i - a^i)$, called the *volume* or *measure* of the interval.

The volume (measure) of the interval I is also denoted $v(I)$ and $\mu(I)$.

Lemma 1. *The measure of an interval in* $\mathbb{R}^n$ *has the following properties.*

a) It is homogeneous, that is, if $\lambda I_{a,b} := I_{\lambda a, \lambda b}$, *where* $\lambda \geq 0$, *then*

$$|\lambda I_{a,b}| = \lambda^n |I_{a,b}| \,.$$

b) It is additive, that is, if the intervals $I, I_1, \dots, I_k$ *are such that* $I = \bigcup\limits_{i=1}^{k} I_i$ *and no two of the intervals* $I_1, \dots, I_k$ *have common interior points, then* $|I| = \sum\limits_{i=1}^{k} |I_i|$.

c) If the interval I *is covered by a finite system of intervals* $I_1, \dots, I_k$, *that is,* $I \subset \bigcup\limits_{i=1}^{k} I_i$, *then* $|I| \leq \sum\limits_{i=1}^{k} |I_i|$.

All these assertions follow easily from Definitions 1 and 2.

b. Partitions of an Interval and a Base in the Set of Partitions Suppose we are given an interval $I = \{x \in \mathbb{R}^n \,|\, a^i \le x^i \le b^i,\, i = 1, \dots, n\}$. Partitions of the coordinate intervals $[a^i, b^i]$, $i = 1, \dots, n$, induce a partition of the interval I into finer intervals obtained as the direct products of the intervals of the partitions of the coordinate intervals.

Definition 3. The representation of the interval I (as the union $I = \bigcup_{j=1}^{k} I_j$ of finer intervals I_j) just described will be called a *partition* of the interval I, and will be denoted by P.

Definition 4. The quantity $\lambda(P) := \max_{1 \le j \le k} d(I_j)$ (the maximum among the diameters of the intervals of the partition P) is called the *mesh* of the partition P.

Definition 5. If in each interval I_j of the partition P we fix a point $\xi_j \in I_j$, we say that we have a *partition with distinguished points.*

The set $\{\xi_1, \dots, \xi_k\}$, as before, will be denoted by the single letter ξ, and the partition with distinguished points by (P, ξ).

In the set $\mathcal{P} = \{(P, \xi)\}$ of partitions with distinguished points on an interval I we introduce the base $\lambda(P) \to 0$ whose elements B_d $(d > 0)$, as in the one-dimensional case, are defined by $B_d := \{(P, \xi) \in \mathcal{P} \,|\, \lambda(P) < d\}$.

The fact that $\mathcal{B} = \{B_d\}$ really is a base follows from the existence of partitions of mesh arbitrarily close to zero.

c. Riemann Sums and the Integral Let $I \to \mathbb{R}$ be a real-valued[1] function on the interval I and $P = \{I_1, \dots, I_k\}$ a partition of this interval with distinguished points $\xi = \{\xi_1, \dots, \xi_k\}$.

Definition 6. The sum

$$\sigma(f, P, \xi) := \sum_{i=1}^{k} f(\xi_i)|I_i|$$

is called the *Riemann sum* of the function f corresponding to the partition of the interval I with distinguished points (P, ξ).

Definition 7. The quantity

$$\int_I f(x)\,\mathrm{d}x := \lim_{\lambda(P) \to 0} \sigma(f, P, \xi)\ ,$$

provided this limit exists, is called the *Riemann integral* of the function f over the interval I.

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[1] Please note that in the following definitions one could assume that the values of f lie in any normed vector space. For example, it might be the space $\mathbb{C}$ of complex numbers or the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$.

We see that this definition, and in general the whole process of constructing the integral over the interval $I \subset \mathbb{R}^n$ is a verbatim repetition of the procedure of defining the Riemann integral over a closed interval of the real line, which is already familiar to us. To highlight the resemblance we have even retained the previous notation $f(x)\,dx$ for the differential form. Equivalent, but more expanded notations for the integral are the following:

$$\int\limits_I f(x^1,\dots,x^n)\,dx^1\cdot\ldots\cdot dx^n \ \text{ or } \ \underbrace{\int\cdots\int\limits_I}_{n} f(x^1,\dots,x^n)\,dx^1\cdot\ldots\cdot dx^n\,.$$

To emphasize that we are discussing an integral over a multidimensional domain I we say that this is a *multiple integral* (double, triple, and so forth, depending on the dimension of I).

d. A necessary Condition for Integrability

Definition 8. If the finite limit in Definition 7 exists for a function $f: I \to \mathbb{R}$, then f is *Riemann integrable* over the interval I.

We shall denote the set of all such functions by $\mathcal{R}(I)$.

We now verify the following elementary necessary condition for integrability.

Proposition 1. *$f \in \mathcal{R}(I) \Rightarrow f$ is bounded on I.*

Proof. Let P be an arbitrary partition of the interval I. If the function f is unbounded on I, then it must be unbounded on some interval I_{i_0} of the partition P. If (P,ξ') and (P,ξ'') are partitions P with distinguished points such that ξ' and ξ'' differ only in the choice of the points ξ'_{i_0} and ξ''_{i_0}, then

$$|\sigma(f,P,\xi') - \sigma(f,P,\xi'')| = |f(\xi'_{i_0}) - f(\xi''_{i_0})|\,|I_{i_0}|\,.$$

By changing one of the points ξ'_{i_0} and ξ''_{i_0}, as a result of the unboundedness of f in I_{i_0}, we could make the right-hand side of this equality arbitrarily large. By the Cauchy criterion, it follows from this that the Riemann sums of f do not have a limit as $\lambda(P) \to 0$. □

11.1.2 The Lebesgue Criterion for Riemann Integrability

When studying the Riemann integral in the one-dimensional case, we acquainted the reader (without proof) with the Lebesgue criterion for the existence of the Riemann integral. We shall now recall certain concepts and prove this criterion.

a. Sets of Measure Zero in $\mathbb{R}^n$

Definition 9. A set $E \subset \mathbb{R}^n$ *has* (n-dimensional) *measure zero* or is a *set of measure zero* (in the Lebesgue sense) if for every $\varepsilon > 0$ there exists a covering of E by an at most countable system $\{I_i\}$ of n-dimensional intervals for which the sum of the volumes $\sum_i |I_i|$ does not exceed ε.

Lemma 2. *a) A point and a finite set of points are sets of measure zero.*

b) The union of a finite or countable number of sets of measure zero is a set of measure zero.

c) A subset of a set of measure zero is itself of measure zero.

d) A nondegenerate[2] *interval $I_{a,b} \subset \mathbb{R}^n$ is not a set of measure zero.*

The proof of Lemma 2 does not differ from the proof of its one-dimensional version considered in Subsect. 6.1.3, paragraph d. Hence we shall not give the details.

Example 1. The set of rational points in $\mathbb{R}^n$ (points all of whose coordinates are rational numbers) is countable and hence is a set of measure zero.

Example 2. Let $f : I \to \mathbb{R}$ be a continuous real-valued function defined on an $(n-1)$-dimensional interval $I \subset \mathbb{R}^{n-1}$. We shall show that its graph in $\mathbb{R}^n$ is a set of n-dimensional measure zero.

Proof. Since the function f is uniformly continuous on I, for $\varepsilon > 0$ we find $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for any two points $x_1, x_2 \in I$ such that $|x_1 - x_2| < \delta$. If we now take a partition P of the interval I with mesh $\lambda(P) < \delta$, then on each interval I_i of this partition the oscillation of the function is less than ε. Hence, if x_i is an arbitrary fixed point of the interval I_i, the n-dimensional interval $\tilde{I}_i = I_i \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon]$ obviously contains the portion of the graph of the function lying over the interval I_i, and the union $\bigcup_i \tilde{I}_i$ covers the whole graph of the function over I. But $\sum_i |\tilde{I}_i| = \sum_i |I_i| \cdot 2\varepsilon = 2\varepsilon|I|$ (here $|I_i|$ is the volume of I_i in $\mathbb{R}^{n-1}$ and $|\tilde{I}_i|$ the volume of $\tilde{I}_i$ in $\mathbb{R}^n$). Thus, by decreasing ε, we can indeed make the total volume of the covering arbitrarily small. $\square$

Remark 1. Comparing assertion b) in Lemma 2 with Example 2, one can conclude that in general the graph of a continuous function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ or a continuous function $f : M \to \mathbb{R}$, where $M \subset \mathbb{R}^{n-1}$, is a set of n-dimensional measure zero in $\mathbb{R}^n$.

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[2] That is, an interval $I_{a,b} = \{x \in \mathbb{R}^n | a^i \le x^i \le b^i, i = 1, \dots, n\}$ such that the strict inequality $a^i < b^i$ holds for each $i \in \{1, \dots, n\}$.

Lemma 3. *a) The class of sets of measure zero remains the same whether the intervals covering the set E in Definition 9, that is, $E \subset \bigcup_i I_i$, are interpreted as an ordinary system of intervals $\{I_i\}$, or in a stricter sense, requiring that each point of the set be an interior point of at least one of the intervals in the covering.*[3]

b) A compact set K in $\mathbb{R}^n$ is a set of measure zero if and only if for every $\varepsilon > 0$ there exists a finite covering of K by intervals the sum of whose volumes is less than ε.

Proof. *a)* If $\{I_i\}$ is a covering of E (that is, $E \subset \bigcup_i I_i$ and $\sum_i |I_i| < \varepsilon$), then, replacing each I_i by a dilation of it from its center, which we denote $\tilde{I}_i$, we obtain a system of intervals $\{\tilde{I}_i\}$ such that $\sum |\tilde{I}_i| < \lambda^n \varepsilon$, where λ is a dilation coefficient that is the same for all intervals. If $\lambda > 1$, it is obvious that the system $\{\tilde{I}_i\}$ will cover E in such a way that every point of E is interior to one of the intervals in the covering.

b) This follows from *a)* and the possibility of extracting a finite covering from any open covering of a compact set K. (The system $\{\tilde{I}_i \setminus \partial \tilde{I}_i\}$ consisting of open intervals obtained from the system $\{\tilde{I}_i\}$ considered in *a)* may serve as such a covering.) □

b. A Generalization of Cantor's Theorem We recall that the oscillation of a function $f : E \to \mathbb{R}$ on the set E has been defined as $\omega(f; E) := \sup\limits_{x_1, x_2 \in E} |f(x_1) - f(x_2)|$, and the oscillation at the point $x \in E$ as $\omega(f; x) := \lim\limits_{\delta \to 0} \omega(f; U_E^\delta(x))$, where $U_E^\delta(x)$ is the δ-neighborhood of x in the set E.

Lemma 4. *If the relation $\omega(f; x) \le \omega_0$ holds at each point of a compact set K for the function $f : K \to \mathbb{R}$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(f; U_K^\delta(x)) < \omega_0 + \varepsilon$ for each point $x \in K$.*

When $\omega_0 = 0$, this assertion becomes Cantor's theorem on uniform continuity of a function that is continuous on a compact set. The proof of Lemma 4 is a verbatim repetition of the proof of Cantor's theorem (Subsect. 6.2.2) and therefore we do not take the time to give it here.

c. Lebesgue's Criterion As before, we shall say that a property holds *at almost all points of a set M* or *almost everywhere on M* if the subset of M where this property does not necessarily hold has measure zero.

Theorem 1. (Lebesgue's criterion). $f \in \mathcal{R}(I) \Leftrightarrow$ (*f is bounded on I*) $\wedge$ (*f is continuous almost everywhere on I*).

[3] In other words, it makes no difference whether we mean closed or open intervals in Definition 9.

Proof. Necessity. If $f \in \mathcal{R}(I)$, then by Proposition 1 the function f is bounded on I. Suppose $|f| \le M$ on I.

We shall now verify that f is continuous at almost all points of I. To do this, we shall show that if the set E of its points of discontinuity does not have measure zero, then $f \notin \mathcal{R}(I)$.

Indeed, representing E in the form $E = \bigcup\limits_{n=1}^{\infty} E_n$, where $E_n = \{x \in I \mid \omega(f;x) \ge 1/n\}$, we conclude from Lemma 2 that if E does not have measure zero, then there exists an index n_0 such that E_{n_0} is also not a set of measure zero. Let P be an arbitrary partition of the interval I into intervals $\{I_i\}$. We break the partition P into two groups of intervals A and B, where

$$A = \Big\{I_i \in P \Big| \, I_i \cap E_{n_0} \neq \varnothing \wedge \omega(f;I_i) \ge \frac{1}{2n_0}\Big\}, \quad \text{and } B = P \setminus A.$$

The system of intervals A forms a covering of the set E_{n_0}. In fact, each point of E_{n_0} lies either in the interior of some interval $I_i \in P$, in which case obviously $I_i \in A$, or on the boundary of several intervals of the partition P. In the latter case, the oscillation of the function must be at least $\frac{1}{2n_0}$ on at least one of these intervals (because of the triangle inequality), and that interval belongs to the system A.

We shall now show that by choosing the set ξ of distinguished points in the intervals of the partition P in different ways we can change the value of the Riemann sum significantly.

To be specific, we choose the sets of points ξ' and ξ'' such that in the intervals of the system B the distinguished points are the same, while in the intervals I_i of the system A, we choose the points ξ_i' and ξ_i'' so that $f(\xi_i') - f(\xi_i'') > \frac{1}{3n_0}$. We then have

$$|\sigma(f,P,\xi') - \sigma(f,P,\xi'')| = \Big|\sum_{I_i \in A} \big(f(\xi_i') - f(\xi_i'')\big)|I_i|\Big| > \frac{1}{3n_0} \sum_{I_i \in A} |I_i| > c > 0\,.$$

The existence of such a constant c follows from the fact that the intervals of the system A form a covering of the set E_{n_0}, which by hypothesis is not a set of measure zero.

Since P was an arbitrary partition of the interval I, we conclude from the Cauchy criterion that the Riemann sums $\sigma(f,P,\xi)$ cannot have a limit as $\lambda(P) \to 0$, that is, $f \notin \mathcal{R}(I)$.

Sufficiency. Let ε be an arbitrary positive number and $E_\varepsilon = \{x \in I \mid \omega(f;x) \ge \varepsilon\}$. By hypothesis, E_ε is a set of measure zero.

Moreover, E_ε is obviously closed in I, so that E_ε is compact. By Lemma 3 there exists a finite system $I_1, \dots, I_k$ of intervals in $\mathbb{R}^n$ such that $E_\varepsilon \subset \bigcup\limits_{i=1}^{k} I_i$ and $\sum\limits_{i=1}^{k} |I_i| < \varepsilon$. Let us set $C_1 = \bigcup\limits_{i=1}^{k} I_i$ and denote by C_2 and C_3 the unions of the intervals obtained from the intervals I_i by dilation with center at the center of I_i and scaling factors 2 and 3 respectively. It is clear that E_ε lies strictly in the interior of C_2 and that the distance d between the boundaries of the sets C_2 and C_3 is positive.

We note that the sum of the volumes of any finite system of intervals lying in C_3, no two of which have any common interior points is at most $3^n\varepsilon$, where n is the dimension of the space $\mathbb{R}^n$. This follows from the definition of the set C_3 and properties of the measure of an interval (Lemma 1).

We also note that any subset of the interval I whose diameter is less than d is either contained in C_3 or lies in the compact set $K = I \setminus (C_2 \setminus \partial C_2)$, where ∂C_2 is the boundary of C_2 (and hence $C_2 \setminus \partial C_2$ is the set of interior points of C_2).

By construction $E_\varepsilon \subset I \setminus K$, so that at every point $x \in K$ we must have $\omega(f;x) < \varepsilon$. By Lemma 4 there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < 2\varepsilon$ for every pair of points $x_1, x_2 \in K$ whose distance from each other is at most δ.

These constructions make it possible now to carry out the proof of the sufficient condition for integrability as follows. We choose any two partitions P' and P'' of the interval I with meshes $\lambda(P')$ and $\lambda(P'')$ less than $\lambda = \min\{d, \delta\}$. Let P be the partition obtained by intersecting all the intervals of the partitions P' and P'', that is, in a natural notation, $P = \{I_{ij} = I'_i \cap I''_j\}$. Let us compare the Riemann sums $\sigma(f, P, \xi)$ and $\sigma(f, P', \xi')$. Taking into account the equality $|I'_i| = \sum\limits_j |I_{ij}|$, we can write

$$|\sigma(f,P,\xi') - \sigma(f,P,\xi)| = \left|\sum_{ij} \left(f(\xi'_i) - f(\xi_{ij})\right)|I_{ij}|\right| \le$$

$$\le \sum\nolimits_1 |f(\xi'_i) - f(\xi_{ij})|\,|I_{ij}| + \sum\nolimits_2 |f(\xi'_i) - f(\xi_{ij})|\,|I_{ij}|\,.$$

Here the first sum $\sum_1$ contains the intervals of the partition P lying in the intervals I'_i of the partition P' contained in the set C_3, and the remaining intervals of P are included in the sum $\sum_2$, that is, they are all necessarily contained in K (after all, $\lambda(P) < d$).

Since $|f| \le M$ on I, replacing $|f(\xi'_i) - f(\xi_{ij})|$ in the first sum by $2M$, we conclude that the first sum does not exceed $2M \cdot 3^n\varepsilon$.

Now, noting that $\xi'_i, \xi_{ij} \in I'_j \subset K$ in the second sum and $\lambda(P') < \delta$, we conclude that $|f(\xi'_i) - f(\xi_{ij})| < 2\varepsilon$, and consequently the second sum does not exceed $2\varepsilon|I|$.

Thus $|\sigma(f, P', \xi') - \sigma(f, P, \xi)| < (2M \cdot 3^n + 2|I|)\varepsilon$, from which (in view of the symmetry between P' and P''), using the triangle inequality, we find that

$$|\sigma(f,P',\xi') - \sigma(f,P'',\xi'')| < 4(3^n M + |I|)\varepsilon$$

for any two partitions P' and P'' with sufficiently small mesh. By the Cauchy criterion we now conclude that $f \in \mathcal{R}(I)$. □

Remark 2. Since the Cauchy criterion for existence of a limit is valid in any complete metric space, the sufficiency part of the Lebesgue criterion (but not the necessity part), as the proof shows, holds for functions with values in any complete normed vector space.

11.1.3 The Darboux Criterion

Let us consider another useful criterion for Riemann integrability of a function, which is applicable only to real-valued functions.

a. Lower and Upper Darboux Sums Let f be a real-valued function on the interval I and $P = \{I_i\}$ a partition of the interval I. We set

$$m_i = \inf_{x \in I_i} f(x), \quad M_i = \sup_{x \in I_i} f(x) .$$

Definition 10. The quantities

$$s(f,P) = \sum_i m_i |I_i| \text{ and } S(f,P) = \sum_i M|I_i|$$

are called the *lower* and *upper Darboux sums* of the function f over the interval I corresponding to the partition P of the interval.

Lemma 5. *The following relations hold between the Darboux sums of a function $f : I \to \mathbb{R}$:*

a) $s(f,P) = \inf_\xi \sigma(f,P,\xi) \le \sigma(f,P,\xi) \le \sup_\xi \sigma(f,P,\xi) = S(f,P)$;

b) if the partition P' of the interval I is obtained by refining intervals of the partition P, then $s(f,P) \le s(f,P') \le S(f,P') \le S(f,P)$;

c) the inequality $s(f,P_1) \le S(f,P_2)$ holds for any pair of partitions P_1 and P_2 of the interval I.

Proof. Relations $a)$ and $b)$ follow immediately from Definitions 6 and 10, taking account, of course, of the definition of the greatest lower bound and least upper bound of a set of numbers.

To prove $c)$ it suffices to consider the auxiliary partition P obtained by intersecting the intervals of the partitions P_1 and P_2. The partition P can be regarded as a refinement of each of the partitions P_1 and P_2, so that $b)$ now implies

$$s(f,P_1) \le s(f,P) \le S(f,P) \le S(f,P_2) . \ \square$$

b. Lower and Upper Integrals

Definition 11. The *lower* and *upper Darboux integrals* of the function $f : I \to \mathbb{R}$ over the interval I are respectively

$$\underline{\mathcal{J}} = \sup_P s(f,P) , \quad \overline{\mathcal{J}} = \inf_P S(f,P) ,$$

where the supremum and infimum are taken over all partitions P of the interval I.

It follows from this definition and the properties of Darboux sums exhibited in Lemma 3 that the inequalities

$$s(f,P) \le \underline{\mathcal{J}} \le \overline{\mathcal{J}} \le S(f,P)$$

hold for any partition P of the interval.

Theorem 2. (Darboux). *For any bounded function* $f : I \to \mathbb{R}$,

$$\Big(\exists \lim_{\lambda(P)\to 0} s(f,P)\Big) \wedge \Big(\lim_{\lambda(P)\to 0} s(f,P) = \underline{\mathcal{J}}\Big) ;$$
$$\Big(\exists \lim_{\lambda(P)\to 0} S(f,P)\Big) \wedge \Big(\lim_{\lambda(F)\to 0} S(f,P) = \overline{\mathcal{J}}\Big) .$$

Proof. If we compare these assertions with Definition 11, it becomes clear that in essence all we have to prove is that the limits exist. We shall verify this for the lower Darboux sums.

Fix $\varepsilon > 0$ and a partition P_ε of the interval I for which $s(f;P_\varepsilon) > \underline{\mathcal{J}} - \varepsilon$. Let Γ_ε be the set of points of the interval I lying on the boundary of the intervals of the partition P_ε. As follows from Example 2, Γ_ε is a set of measure zero. Because of the simple structure of Γ_ε, it is even obvious that there exists a number λ_ε such that the sum of the volumes of those intervals that intersect Γ_ε is less than ε for every partition P such that $\lambda(P) < \lambda_\varepsilon$.

Now taking any partition P with mesh $\lambda(P) < \lambda_\varepsilon$, we form an auxiliary partition P' obtained by intersecting the intervals of the partitions P and P_ε. By the choice of the partition P_ε and the properties of Darboux sums (Lemma 5), we find

$$\underline{\mathcal{J}} - \varepsilon < s(f,P_\varepsilon) < s(f,P') \le \underline{\mathcal{J}} .$$

We now remark that the sums $s(f,P')$ and $s(f,P)$ both contain all the terms that correspond to intervals of the partition P that do not meet Γ_ε. Therefore, if $|f(x)| \le M$ on I, then

$$|s(f,P') - s(f,P)| < 2M\varepsilon$$

and taking account of the preceding inequalities, we thereby find that for $\lambda(P) < \lambda_\varepsilon$ we have the relation

$$\underline{\mathcal{J}} - s(f,P) < (2M+1)\varepsilon .$$

Comparing the relation just obtained with Definition 11, we conclude that the limit $\lim\limits_{\lambda(P)\to 0} s(f,P)$ does indeed exist and is equal to $\underline{\mathcal{J}}$.

Similar reasoning can be carried out for the upper sums. □

c. The Darboux Criterion for Integrability of a Real-valued Function

Theorem 3. (The Darboux criterion). *A real-valued function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}^n$ is integrable over that interval if and only if it is bounded on I and its upper and lower Darboux integrals are equal.*

Thus,

$$f \in \mathcal{R}(I) \iff (f \text{ is bounded on } I) \wedge (\underline{\mathcal{J}} = \overline{\mathcal{J}}) .$$

Proof. Necessity. If $f \in \mathcal{R}(I)$, then by Proposition 1 the function f is bounded on I. It follows from Definition 7 of the integral, Definition 11 of the quantities $\underline{\mathcal{J}}$ and $\overline{\mathcal{J}}$, and part $a)$ of Lemma 5 that in this case $\underline{\mathcal{J}} = \overline{\mathcal{J}}$.

Sufficiency. Since $s(f,P) \leq \sigma(f,P,\xi) \leq S(f,P)$ when $\underline{\mathcal{J}} = \overline{\mathcal{J}}$, the extreme terms in these inequalities tend to the same limit by Theorem 2 as $\lambda(P) \to 0$. Therefore $\sigma(f,P,\xi)$ has the same limit as $\lambda(P) \to 0$. □

Remark 3. It is clear from the proof of the Darboux criterion that if a function is integrable, its lower and upper Darboux integrals are equal to each other and to the integral of the function.

11.1.4 Problems and Exercises

1. a) Show that a set of measure zero has no interior points.

b) Show that not having interior points by no means guarantees that a set is of measure zero.

c) Construct a set having measure zero whose closure is the entire space $\mathbb{R}^n$.

d) A set $E \subset I$ is said to have content zero if for every $\varepsilon > 0$ it can be covered by a finite system of intervals $I_1, \ldots, I_k$ such that $\sum\limits_{i=1}^{k} |I_i| < \varepsilon$. Is every bounded set of measure zero a set of content zero?

Show that if a set $E \subset \mathbb{R}^n$ is the direct product $\mathbb{R} \times e$ of the line $\mathbb{R}$ and a set $e \subset \mathbb{R}^{n-1}$ of $(n-1)$-dimensional measure zero, then E is a set of n-dimensional measure zero.

2. a) Construct the analogue of the Dirichlet function in $\mathbb{R}^n$ and show that a bounded function $f : I \to \mathbb{R}$ equal to zero at almost every point of the interval I may still fail to belong to $\mathcal{R}(I)$.

b) Show that if $f \in \mathcal{R}(I)$ and $f(x) = 0$, at almost all points of the interval I, then $\int\limits_I f(x)\,dx = 0$.

3. There is a small difference between our earlier definition of the Riemann integral on a closed interval $I \subset \mathbb{R}$ and Definition 7 for the integral over an interval of arbitrary dimension. This difference involves the definition of a partition and the measure of an interval of the partition. Clarify this nuance for yourself and verify that

$$\int_a^b f(x)\,dx = \int_I f(x)\,dx\,, \quad \text{if } a < b$$

and

$$\int_a^b f(x)\,dx = -\int_I f(x)\,dx\,, \quad \text{if } a > b\,,$$

where I is the interval on the real line $\mathbb{R}$ with endpoints a and b.

4. a) Prove that a real-valued function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}^n$ is integrable over that interval if and only if for every $\varepsilon > 0$ there exists a partition P of I such that $S(f;P) - s(f;P) < \varepsilon$.

b) Using the result of a) and assuming that we are dealing with a real-valued function $f : I \to \mathbb{R}$, one can simplify slightly the proof of the sufficiency of the Lebesgue criterion. Try to carry out this simplification by yourself.

11.2 The Integral over a Set

11.2.1 Admissible Sets

In what follows we shall be integrating functions not only over an interval, but also over other sets in $\mathbb{R}^n$ that are not too complicated.

Definition 1. A set $E \subset \mathbb{R}^n$ is *admissible* if it is bounded in $\mathbb{R}^n$ and its boundary is a set of measure zero (in the sense of Lebesgue).

Example 1. A cube, a tetrahedron, and a ball in $\mathbb{R}^3$ (or $\mathbb{R}^n$) are admissible sets.

Example 2. Suppose the functions $\varphi_i : I \to \mathbb{R}$, $i = 1, 2$, defined on an $(n-1)$-dimensional interval $I \subset \mathbb{R}^n$ are such that $\varphi_1(x) < \varphi_2(x)$ at every point $x \in I$. If these functions are continuous, Example 2 of Sect. 11.1 makes it possible to assert that the domain in $\mathbb{R}^n$ bounded by the graphs of these functions and the cylindrical lateral surface lying over the boundary ∂I of I is an admissible set in $\mathbb{R}^n$.

We recall that the boundary ∂E of a set $E \subset \mathbb{R}^n$ consists of the points x such that every neighborhood of x contains both points of E and points of the complement of E in $\mathbb{R}^n$. Hence we have the following lemma.

Lemma 1. *For any sets $E, E_1, E_2 \subset \mathbb{R}^n$, the following assertions hold:*

a) ∂E *is a closed subset of* $\mathbb{R}^n$;

b) $\partial(E_1 \cup E_2) \subset \partial E_1 \cup \partial E_2$;

c) $\partial(E_1 \cap E_2) \subset \partial E_1 \cup \partial E_2$;

d) $\partial(E_1 \setminus E_2) \subset \partial E_1 \cup \partial E_2$.

This lemma and Definition 1 together imply the following lemma.

Lemma 2. *The union or intersection of a finite number of admissible sets is an admissible set; the difference of admissible sets is also an admissible set.*

Remark 1. For an infinite collection of admissible sets Lemma 2 is generally not true, and the same is the case with assertions b) and c) of Lemma 1.

Remark 2. The boundary of an admissible set is not only closed, but also bounded in $\mathbb{R}^n$, that is, it is a compact subset of $\mathbb{R}^n$. Hence by Lemma 3 of Sect. 11.1, it can even be covered by a finite set of intervals whose total content (volume) is arbitrarily close to zero.

We now consider the characteristic function

$$\chi_E(x) = \begin{cases} 1\,, & \text{if } x \in E\,, \\ 0\,, & \text{if } x \notin E\,, \end{cases}$$

of an admissible set E. Like the characteristic function of any set E, the function $\chi_E(x)$ has discontinuities at the boundary points of the set E and at no other points. Hence if E is an admissible set, the function $\chi_E(x)$ is continuous at almost all points of $\mathbb{R}^n$.

11.2.2 The Integral over a Set

Let f be a function defined on a set E. We shall agree, as before, to denote the function equal to $f(x)$ for $x \in E$ and to 0 outside E by $f\chi_E(x)$ (even though f may happen to be undefined outside of E).

Definition 2. The *integral of f over E* is given by

$$\int\limits_E f(x)\,\mathrm{d}x := \int\limits_{I \supset E} f\chi_E(x)\,\mathrm{d}x\,,$$

where I is any interval containing E.

If the integral on the right-hand side of this equality does not exist, we say that f is (Riemann) *nonintegrable over E*. Otherwise f is (Riemann) *integrable over E*.

The set of all functions that are Riemann integrable over E will be denoted $\mathcal{R}(E)$.

Definition 2 of course requires some explanation, which is provided by the following lemma.

Lemma 3. *If I_1 and I_2 are two intervals, both containing the set E, then the integrals*

$$\int_{I_1} f\chi_E(x)\,\mathrm{d}x \quad \text{and} \quad \int_{I_2} f\chi_E(x)\,\mathrm{d}x$$

either both exist or both fail to exist, and in the first case their values are the same.

Proof. Consider the interval $I = I_1 \cap I_2$. By hypothesis $I \supset E$. The points of discontinuity of $f\chi_E$ are either points of discontinuity of f on E, or the result of discontinuities of χ_E, in which case they lie on ∂E. In any case, all these points lie in $I = I_1 \cap I_2$. By Lebesgue's criterion (Theorem 1 of Sect. 11.1) it follows that the integrals of $f\chi_E$ over the intervals I, I_1, and I_2 either all exist or all fail to exist. If they do exist, we may choose partitions of I, I_1, and I_2 to suit ourselves. Therefore we shall choose only those partitions of I_1 and I_2 obtained as extensions of partitions of $I = I_1 \cap I_2$. Since the function is zero outside I, the Riemann sums corresponding to these partitions of I_1 and I_2 reduce to Riemann sums for the corresponding partition of I. It then results from passage to the limit that the integrals over I_1 and I_2 are equal to the integral of the function in question over I. □

Lebesgue's criterion (Theorem 1 of Sect. 11.1) for the existence of the integral over an interval and Definition 2 now imply the following theorem.

Theorem 1. *A function $f : E \to \mathbb{R}$ is integrable over an admissible set if and only if it is bounded and continuous at almost all points of E.*

Proof. Compared with f, the function $f\chi_E$ may have additional points of discontinuity only on the boundary ∂E of E, which by hypothesis is a set of measure zero. □

11.2.3 The Measure (Volume) of an Admissible Set

Definition 3. The (Jordan) *measure* or *content of* a bounded *set* $E \subset \mathbb{R}^n$ is

$$\mu(E) := \int_E 1 \cdot \mathrm{d}x ,$$

provided this Riemann integral exists.

Since

$$\int_E 1 \cdot \mathrm{d}x = \int_{I \supset E} \chi_E(x)\,\mathrm{d}x ,$$

and the discontinuities of χ_E form the set ∂E, we find by Lebesgue's criterion that the measure just introduced is defined only for admissible sets.

Thus admissible sets, and only admissible sets, are measurable in the sense of Definition 3.

Let us now ascertain the geometric meaning of $\mu(E)$. If E is an admissible set then

$$\mu(E) = \int_{I \supset E} \chi_E(x)\,dx = \underline{\int}_{I \supset E} \chi_E(x)\,dx = \overline{\int}_{I \supset E} \chi_E(x)\,dx\,,$$

where the last two integrals are the upper and lower Darboux integrals respectively. By the Darboux criterion for existence of the integral (Theorem 3) the measure $\mu(E)$ of a set is defined if and only if these lower and upper integrals are equal. By the theorem of Darboux (Theorem 2 of Sect. 11.1) they are the limits of the upper and lower Darboux sums of the function χ_E corresponding to partitions P of I. But by definition of χ_E the lower Darboux sum is the sum of the volumes of the intervals of the partition P that are entirely contained in E (the volume of a polyhedron inscribed in E), while the upper sum is the sum of the volumes of the intervals of P that intersect E (the volume of a circumscribed polyhedron). Hence $\mu(E)$ is the common limit as $\lambda(P) \to 0$ of the volumes of polyhedra inscribed in and circumscribed about E, in agreement with the accepted idea of the volume of simple solids $E \subset \mathbb{R}^n$.

For $n = 1$ content is usually called *length*, and for $n = 2$ it is called *area*.

Remark 3. Let us now explain why the measure $\mu(E)$ introduced in Definition 3 is sometimes called Jordan measure.

Definition 4. A set $E \subset \mathbb{R}^n$ is a *set of measure zero in the sense of Jordan* or a *set of content zero* if for every $\varepsilon > 0$ it can be covered by a finite system of intervals $I_1, \dots, I_k$ such that $\sum_{i=1}^{k} |I_i| < \varepsilon$.

Compared with measure zero in the sense of Lebesgue, a requirement that the covering be finite appears here, shrinking the class of sets of Lebesgue measure zero. For example, the set of rational points is a set of measure zero in the sense of Lebesgue, but not in the sense of Jordan.

In order for the least upper bound of the contents of polyhedra inscribed in a bounded set E to be the same as the greatest lower bound of the contents of polyhedra circumscribed about E (and to serve as the measure $\mu(E)$ or content of E), it is obviously necessary and sufficient that the boundary ∂E of E have measure 0 in the sense of Jordan. That is the motivation for the following definition.

Definition 5. A set E is *Jordan-measurable* if it is bounded and its boundary has Jordan measure zero.

As Remark 2 shows, the class of Jordan-measurable subsets is precisely the class of admissible sets introduced in Definition 1. That is the reason the measure $\mu(E)$ defined earlier can be called (and is called) the *Jordan measure* of the (Jordan-measurable) set E.

11.2.4 Problems and Exercises

1. a) Show that if a set $E \subset \mathbb{R}^n$ is such that $\mu(E) = 0$, then the relation $\mu(\overline{E}) = 0$ also holds for the closure $\overline{E}$ of the set.

b) Give an example of a bounded set E of Lebesgue measure zero whose closure $\overline{E}$ is not a set of Lebesgue measure zero.

c) Determine whether assertion b) of Lemma 3 in Sect. 11.1 should be understood as asserting that the concepts of Jordan measure zero and Lebesgue measure zero are the same for compact sets.

d) Prove that if the projection of a bounded set $E \subset \mathbb{R}^n$ onto a hyperplane $\mathbb{R}^{n-1}$ has $(n-1)$-dimensional measure zero, then the set E itself has n-dimensional measure zero.

e) Show that a Jordan-measurable set whose interior is empty has measure 0.

2. a) Is it possible for the integral of a function f over a bounded set E, as introduced in Definition 2, to exist if E is not an admissible (Jordan-measurable) set?

b) Is a constant function $f : E \to \mathbb{R}$ integrable over a bounded but Jordan-nonmeasurable set E?

c) Is it true that if a function f is integrable over E, then the restriction $f\big|A$ of this function to any subset $A \subset E$ is integrable over A?

d) Give necessary and sufficient conditions on a function $f : E \to \mathbb{R}$ defined on a bounded (but not necessarily Jordan-measurable) set E under which the Riemann integral of f over E exists.

3. a) Let E be a set of Lebesgue measure 0 and $f : E \to \mathbb{R}$ a bounded continuous function on E. Is f always integrable on E?

b) Answer question a) assuming that E is a set of Jordan measure zero.

c) What is the value of the integral of the function f in a) if it exists?

4. *The Brunn–Minkowski inequality.* Given two nonempty sets $A, B \subset \mathbb{R}^n$, we form their (vector) sum in the sense of Minkowski $A + B := \{a + b \mid a \in A,\, b \in B\}$. Let $V(E)$ denote the content of a set $E \subset \mathbb{R}^n$.

a) Verify that if A and B are standard n-dimensional intervals (parallelepipeds), then

$$V^{1/n}(A+B) \geq V^{1/n}(A) + V^{1/n}(B) .$$

b) Now prove the preceding inequality (the *Brunn–Minkowski inequality*) for arbitrary measurable compact sets A and B.

c) Show that equality holds in the Brunn–Minkowski inequality only in the following three cases: when $V(A+B) = 0$, when A and B are singleton (one-point) sets, and when A and B are similar convex sets.

11.3 General Properties of the Integral

11.3.1 The Integral as a Linear Functional

Proposition 1. *a) The set $\mathcal{R}(E)$ of functions that are Riemann-integrable over a bounded set $E \subset \mathbb{R}^n$ is a vector space with respect to the standard operations of addition of functions and multiplication by constants.*

b) The integral is a linear functional

$$\int_E : \mathcal{R}(E) \to \mathbb{R} \ \textit{ on the set } \mathcal{R}(E) .$$

Proof. Noting that the union of two sets of measure zero is also a set of measure zero, we see that assertion a) follows immediately from the definition of the integral and the Lebesgue criterion for existence of the integral of a function over an interval.

Taking account of the linearity of Riemann sums, we obtain the linearity of the integral by passage to the limit. □

Remark 1. If we recall that the limit of the Riemann sums as $\lambda(P) \to 0$ must be the same independently of the set of distinguished points ξ, we can conclude that

$$\big(f \in \mathcal{R}(E)\big) \bigwedge \big(f(x) = 0 \text{ almost everywhere on } E\big) \Longrightarrow \Big(\int_E f(x)\,\mathrm{d}x = 0\Big) .$$

Therefore, if two integrable functions are equal at almost all points of E, then their integrals over E are also equal. Hence if we pass to the quotient space of $\mathcal{R}(E)$ obtained by identifying functions that are equal at almost all points of E, we obtain a vector space $\widetilde{\mathcal{R}}(E)$ on which the integral is also a linear function.

11.3.2 Additivity of the Integral

Although we shall always be dealing with admissible sets $E \subset \mathbb{R}^n$, this assumption was dispensable in Subsect. 11.3.1 (and we dispensed with it). From now on we shall be talking only of admissible sets.

Proposition 2. *Let E_1 and E_2 be admissible sets in $\mathbb{R}^n$ and f a function defined on $E_1 \cup E_2$.*

a) The following relations hold:

$$\Big(\exists \int_{E_1 \cup E_2} f(x)\,\mathrm{d}x\Big) \Longleftrightarrow \Big(\exists \int_{E_1} f(x)\,\mathrm{d}x\Big) \bigwedge \Big(\exists \int_{E_2} f(x)\,\mathrm{d}x\Big) \Longrightarrow \exists \int_{E_1 \cap E_2} f(x)\,\mathrm{d}x .$$

b) If in addition it is known that $\mu(E_1 \cap E_2) = 0$, the following equality holds when the integrals exist:

$$\int_{E_1 \cup E_2} f(x)\,\mathrm{d}x = \int_{E_1} f(x)\,\mathrm{d}x + \int_{E_2} f(x)\,\mathrm{d}x\,.$$

Proof. Assertion *a*) follows from Lebesgue's criterion for existence of the Riemann integral over an admissible set (Theorem 1 of Sect. 11.2). Here it is only necessary to recall that the union and intersection of admissible sets are also admissible sets (Lemma 2 of Sect. 11.2).

To prove *b*) we begin by remarking that

$$\chi_{E_1 \cup E_2} = \chi_{E_1}(x) + \chi_{E_2}(x) - \chi_{E_1 \cap E_2}(x)\,.$$

Therefore,

$$\int_{E_1 \cup E_2} f(x)\,\mathrm{d}x = \int_{I \supset E_1 \cup E_2} f\chi_{E_1 \cup E_2}(x)\,\mathrm{d}x =$$

$$= \int_I f\chi_{E_1}(x)\,\mathrm{d}x + \int_I f\chi_{E_2}(x)\,\mathrm{d}x - \int_I \chi_{E_1 \cap E_2}(x)\,\mathrm{d}x =$$

$$= \int_{E_1} f(x)\,\mathrm{d}x + \int_{E_2} f(x)\,\mathrm{d}x\,.$$

The essential point is that the integral

$$\int_I f\chi_{E_1 \cap E_2}(x)\,\mathrm{d}x = \int_{E_1 \cap E_2} f(x)\,\mathrm{d}x\,,$$

as we know from part *a*), exists; and since $\mu(E_1 \cap E_2) = 0$, it equals zero (see Remark 1). □

11.3.3 Estimates for the Integral

a. A General Estimate We begin with a general estimate of the integral that is also valid for functions with values in any complete normed space.

Proposition 3. *If $f \in \mathcal{R}(E)$, then $|f| \in \mathcal{R}(E)$, and the inequality*

$$\left|\int_E f(x)\,\mathrm{d}x\right| \le \int_E |f|(x)\,\mathrm{d}x$$

holds.

Proof. The relation $|f| \in \mathcal{R}(E)$ follows from the definition of the integral over a set and the Lebesgue criterion for integrability of a function over an interval.

The inequality now follows from the corresponding inequality for Riemann sums and passage to the limit. □

b. The Integral of a Nonnegative Function The following propositions apply only to real-valued functions.

Proposition 4. *The following implication holds for a function* $f : E \to \mathbb{R}$ *:*

$$\big(f \in \mathcal{R}(E)\big) \wedge \big(\forall x \in E\ (f(x) \geq 0)\big) \Longrightarrow \int_E f(x)\,\mathrm{d}x \geq 0\,.$$

Proof. Indeed, if $f(x) \geq 0$ on E, then $f\chi_E(x) \geq 0$ in $\mathbb{R}^n$. Then, by definition,

$$\int_E f(x)\,\mathrm{d}x = \int_{I\supset E} f\chi_E(x)\,\mathrm{d}x\,.$$

This last integral exists by hypothesis. But it is the limit of nonnegative Riemann sums and hence nonnegative. □

From Proposition 4 just proved, we obtain successively the following corollaries.

Corollary 1.

$$\big(f, g \in \mathcal{R}(E)\big) \wedge \big(f \leq g \ \ on\ E\big) \Longrightarrow \left(\int_E f(x)\,\mathrm{d}x \leq \int_E g(x)\,\mathrm{d}x\right)\,.$$

Corollary 2. *If* $f \in \mathcal{R}(E)$ *and the inequalities* $m \leq f(x) \leq M$ *hold at every point of the admissible set* E, *then*

$$m\mu(E) \leq \int_E f(x)\,\mathrm{d}x \leq M\mu(E)\,.$$

Corollary 3. *If* $f \in \mathcal{R}(E)$, $m = \inf\limits_{x\in E} f(x)$, *and* $M = \sup\limits_{x\in E} f(x)$, *then there is a number* $\theta \in [m, M]$ *such that*

$$\int_E f(x)\,\mathrm{d}x = \theta\mu(E)\,.$$

Corollary 4. *If* E *is a connected admissible set and the function* $f : E \to \mathbb{R}$ *is continuous, then there exists a point* $\xi \in E$ *such that*

$$\int_E f(x)\,\mathrm{d}x = f(\xi)\mu(E)\,.$$

Corollary 5. *If in addition to the hypotheses of Corollary 2 the function* $g \in \mathcal{R}(E)$ *is nonnegative on* E, *then*

$$m\int_E g(x)\,\mathrm{d}x \leq \int_E fg(x)\,\mathrm{d}x \leq M\int_E g(x)\,\mathrm{d}x\,.$$

Corollary 4 is a generalization of the one-dimensional result and is usually called by the same name, that is, the *mean-value theorem for the integral*.

Proof. Corollary 5 follows from the inequalities $mg(x) \le f(x)g(x) \le Mg(x)$ taking account of the linearity of the integral and Corollary 1. It can also be proved directly by passing from integrals over E to the corresponding integrals over an interval, verifying the inequalities for the Riemann sums, and then passing to the limit. Since all these arguments were carried out in detail in the one-dimensional case, we shall not give the details. We note merely that the integrability of the product $f \cdot g$ of the functions f and g obviously follows from Lebesgue's criterion. □

We shall now illustrate these relations in practice, using them to verify the following very useful lemma.

Lemma. *a) If the integral of a nonnegative function $f : I \to \mathbb{R}$ over the interval I equals zero, then $f(x) = 0$ at almost all points of the interval I.*

b) Assertion a) remains valid if the interval I in it is replaced by any admissible (Jordan-measurable) set E.

Proof. By Lebesgue's criterion the function $f \in \mathcal{R}(E)$ is continuous at almost all points of the interval I. For that reason the proof of $a)$ will be achieved if we show that $f(a) = 0$ at each point of continuity $a \in I$ of the function f.

Assume that $f(a) > 0$. Then $f(x) \ge c > 0$ in some neighborhood $U_I(a)$ of a (the neighborhood may be assumed to be an interval). Then, by the properties of the integral just proved,

$$\int\limits_I f(x)\,\mathrm{d}x = \int\limits_{U_I(a)} f(x)\,\mathrm{d}x + \int\limits_{I\setminus U_I(a)} f(x)\,\mathrm{d}x \ge \int\limits_{U_I(a)} f(x)\,\mathrm{d}x \ge c\mu\big(U_I(a)\big) > 0\,.$$

This contradiction verifies assertion $a)$. If we apply this assertion to the function $f\chi_E$ and take account of the relation $\mu(\partial E) = 0$, we obtain assertion $b)$. □

Remark 2. It follows from the lemma just proved that if E is a Jordan-measurable set in $\mathbb{R}^n$ and $\widetilde{\mathcal{R}}(E)$ is the vector space considered in Remark 1, consisting of equivalence classes of functions that are integrable over E and differ only on sets of Lebesgue measure zero, then the quantity $\|f\| = \int\limits_E |f|(x)\,\mathrm{d}x$ is a norm on $\widetilde{\mathcal{R}}(E)$.

Proof. Indeed, the inequality $\int\limits_E |f|(x)\,\mathrm{d}x = 0$ now implies that f is in the same equivalence class as the identically zero function. □

11.3.4 Problems and Exercises

1. Let E be a Jordan-measurable set of nonzero measure, $f : E \to \mathbb{R}$ a continuous nonnegative integrable function on E, and $M = \sup_{x \in E} f(x)$. Prove that

$$\lim_{n\to\infty}\left(\int_E f^n(x)\,dx\right)^{1/n} = M .$$

2. Prove that if $f, g \in \mathcal{R}(E)$, then the following are true.

a) Hölder's inequality

$$\left|\int_E (f\cdot g)(x)\,dx\right| \le \left(\int_E |f|^p(x)\,dx\right)^{1/p}\left(\int_E |g|^q(x)\,dx\right)^{1/q},$$

where $p \ge 1$, $q \ge 1$, and $\frac{1}{p}+\frac{1}{q}=1$;

b) Minkowski's inequality

$$\left(\int_E |f+g|^p\,dx\right)^{1/p} \le \left(\int_E |f|^p(x)\,dx\right)^{1/p} + \left(\int_E |g|^p(x)\,dx\right)^{1/p},$$

if $p \ge 1$.

Show that

c) the preceding inequality reverses direction if $0 < p < 1$;

d) equality holds in Minkowski's inequality if and only if there exists $\lambda \ge 0$ such that one of the equalities $f = \lambda g$ or $g = \lambda f$ holds except on a set of measure zero in E;

e) the quantity $\|f\|_p = \left(\frac{1}{\mu(E)}\int_E |f|^p(x)\,dx\right)^{1/p}$, where $\mu(E) > 0$, is a monotone function of $p \in \mathbb{R}$ and is a norm on the space $\widetilde{\mathcal{R}}(E)$ for $p \ge 1$.

Find the conditions under which equality holds in Hölder's inequality.

3. Let E be a Jordan-measurable set in $\mathbb{R}^n$ with $\mu(E) > 0$. Verify that if $\varphi \in C(E,\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{1}{\mu(E)}\int_E \varphi(x)\,dx\right) \le \frac{1}{\mu(E)}\int_E (f\circ\varphi)(x)\,dx .$$

4. *a*) Show that if E is a Jordan-measurable set in $\mathbb{R}^n$ and the function $f : E \to \mathbb{R}$ is integrable over E and continuous at an interior point $a \in E$, then

$$\lim_{\delta\to+0}\frac{1}{\mu\left(U_E^\delta(a)\right)}\int_{U_E^\delta(a)} f(x)\,dx = f(a),$$

where, as usual, $U_E^\delta(a)$ is the δ-neighborhood of the point in E.

b) Verify that the preceding relation remains valid if the condition that a is an interior point of E is replaced by the condition $\mu\left(U_E^\delta(a)\right) > 0$ for every $\delta > 0$.

11.4 Reduction of a Multiple Integral to an Iterated Integral

11.4.1 Fubini's Theorem

Up to now, we have discussed only the definition of the integral, the conditions under which it exists, and its general properties. In the present section we shall prove Fubini's theorem,[4] which, together with the formula for change of variable, is a tool for computing multiple integrals.

Theorem.[5] *Let $X \times Y$ be an interval in $\mathbb{R}^{m+n}$, which is the direct product of intervals $X \subset \mathbb{R}^m$ and $Y \subset R^n$. If the function $f : X \times Y \to \mathbb{R}$ is integrable over $X \times Y$, then all three of the integrals*

$$\boxed{\int\limits_{X\times Y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,, \quad \int\limits_X \mathrm{d}x \int\limits_Y f(x,y)\,\mathrm{d}y\,, \quad \int\limits_Y \mathrm{d}y \int\limits_X f(x,y)\,\mathrm{d}x}$$

exist and are equal.

Before taking up the proof of this theorem, let us decode the meaning of the symbolic expressions that occur in the statement of it. The integral $\int\limits_{X\times Y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ is the integral of the function f over the set $X \times Y$, which we are familiar with, written in terms of the variables $x \in X$ and $y \in Y$. The iterated integral $\int\limits_X \mathrm{d}x \int\limits_Y f(x,y)\,\mathrm{d}y$ should be understood as follows: For each fixed $x \in X$ the integral $F(x) = \int\limits_Y f(x,y)\,\mathrm{d}y$ is computed, and the resulting function $F : X \to \mathbb{R}$ is then to be integrated over X. If, in the process, the integral $\int\limits_Y f(x,y)\,\mathrm{d}y$ does not exist for some $x \in X$, then $F(x)$ is set equal to any value between the lower and upper Darboux integrals $\underline{\mathcal{J}}(x) = \underline{\int\limits_Y} f(x,y)\,\mathrm{d}y$ and $\overline{\mathcal{J}}(x) = \overline{\int\limits_Y} f(x,y)\,\mathrm{d}y$, including the upper and lower integrals $\underline{\mathcal{J}}(x)$ and $\overline{\mathcal{J}}(x)$ themselves. It will be shown that in that case $F \in \mathcal{R}(X)$. The iterated integral $\int$ $limits_Y$ $\mathrm{d}y \int\limits_X f(x,y)\,\mathrm{d}x$ has a similar meaning.

It will become clear in the course of the proof that the set of values of $x \in X$ at which $\underline{\mathcal{J}}(x) \neq \overline{\mathcal{J}}(x)$ is a set of m-dimensional measure zero in X.

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[4] G. Fubini (1870–1943) – Italian mathematician. His main work was in the area of the theory of functions and geometry.

[5] This theorem was proved long before the theorem known in analysis as Fubini's theorem, of which it is a special case. However, it has become the custom to refer to theorems making it possible to reduce the computation of multiple integrals to iterated integrals in lower dimensions as theorems of Fubini type, or, for brevity, Fubini's theorem.

Similarly, the set of $y \in Y$ at which the integral $\int\limits_X f(x,y)\,\mathrm{d}x$ may fail to exist will turn out to be a set of n-dimensional measure zero in Y.

We remark finally that, in contrast to the integral over an $(m+n)$-dimensional interval, which we previously agreed to call a *multiple integral*, the successively computed integrals of the function $f(x,y)$ over Y and then over X or over X and then over Y are customarily called *iterated integrals* of the function.

If X and Y are closed intervals on the line, the theorem stated here theoretically reduces the computation of a double integral over the interval $X \times Y$ to the successive computation of two one-dimensional integrals. It is clear that by applying this theorem several times, one can reduce the computation of an integral over a k-dimensional interval to the successive computation of k one-dimensional integrals.

The essence of the theorem we have stated is very simple and consists of the following. Consider a Riemann sum $\sum\limits_{i,j} f(x_i,y_j)|X_i|\cdot|Y_j|$ corresponding to a partition of the interval $X \times Y$ into intervals $X_i \times Y_j$. Since the integral over the interval $X \times Y$ exists, the distinguished points ξ_{ij} can be chosen as we wish, and we choose them as the "direct product" of choices $x_i \in X_i \subset X$ and $y_j \in Y_j \in Y$. We can then write

$$\sum_{i,j} f(x_i,y_j)|X_i|\cdot|Y_j| = \sum_i |X_i| \sum_j f(x_i,y_j)|Y_j| = \sum_j |Y_j| \sum_i f(x_i,y_j)|X_j|\,,$$

and this is the prelimit form of theorem.

We now give the formal proof.

Proof. Every partition P of the interval $X \times Y$ is induced by corresponding partitions P_X and P_Y of the intervals X and Y. Here every interval of the partition P is the direct product $X_i \times Y_j$ of certain intervals X_i and Y_j of the partitions P_X and P_Y respectively. By properties of the volume of an interval we have $|X_i \times Y_j| = |X_i|\cdot|Y_j|$, where each of these volumes is computed in the space $\mathbb{R}^{m+n}$, $\mathbb{R}^m$, or $\mathbb{R}^n$ in which the interval in question is situated.

Using the properties of the greatest lower bound and least upper bound and the definition of the lower and upper Darboux sums and integrals, we now carry out the following estimates:

$$s(f,P) = \sum_{i,j} \inf_{\substack{x\in X_i\\ y\in Y_j}} f(x,y)|X_i \times Y_j| \le \sum_i \inf_{x\in X_i}\Big(\sum_j \inf_{y\in Y_j} f(x,y)|Y_j|\Big)|X_i| \le$$

$$\le \sum_i \inf_{x\in X_i}\Big(\underline{\int\limits_Y} f(x,y)\,\mathrm{d}y\Big)|X_i| \le \sum_i \inf_{x\in X_i} F(x)|X_i| \le$$

$$\le \sum_i \sup_{x\in X_i} F(x)|X_i| \le \sum_i \sup_{x\in X_i}\Big(\overline{\int\limits_Y} f(x,y)\,\mathrm{d}y\Big)|X_i| \le$$

$$\leq \sum_i \sup_{x \in X_i} \left(\sum_j \sup_{y \in Y_j} F(x,y)|Y_j| \right) |X_i| \leq$$
$$\leq \sum_{i,j} \sup_{\substack{x \in X_i \\ y \in Y_j}} f(x,y)|X_i \times Y_j| = S(f,P) .$$

Since $f \in \mathcal{R}(X \times Y)$, both of the extreme terms in these inequalities tend to the value of the integral of the function over the interval $X \times Y$ as $\lambda(P) \to 0$. This fact enables us to conclude that $F \in \mathcal{R}(X)$ and that the following equality holds:

$$\int\limits_{X \times Y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_X F(x)\,\mathrm{d}x .$$

We have carried out the proof for the case when the iterated integration is carried out first over Y, then over X. It is clear that similar reasoning can be used in the case when the integration over X is done first. □

11.4.2 Some Corollaries

Corollary 1. *If $f \in \mathcal{R}(X \times Y)$, then for almost all $x \in X$ (in the sense of Lebesgue) the integral $\int\limits_Y f(x,y)\,\mathrm{d}y$ exists, and for almost all $y \in Y$ the integral $\int\limits_X f(x,y)\,\mathrm{d}x$ exists.*

Proof. By the theorem just proved,

$$\int\limits_X \left(\overline{\int\limits_Y} f(x,y)\,\mathrm{d}y - \int\limits_{\underline{Y}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = 0 .$$

But the difference of the upper and lower integrals in parentheses is non-negative. We can therefore conclude by the lemma of Sect. 11.3 that this difference equals zero at almost all points $x \in X$.

Then by the Darboux criterion (Theorem 3 of Sect. 11.1) the integral $\int\limits_Y f(x,y)\,\mathrm{d}y$ exists for almost all values of $x \in X$.

The second half of the corollary is proved similarly. □

Corollary 2. *If the interval $I \subset \mathbb{R}^n$ is the direct product of the closed intervals $I_i = [a^i, b^i]$, $i = 1, \dots, n$, then*

$$\int\limits_I f(x)\,\mathrm{d}x = \int\limits_{a^n}^{b^n} \mathrm{d}x^n \int\limits_{a^{n-1}}^{b^{n-1}} \mathrm{d}x^{n-1} \cdots \int\limits_{a^1}^{b^1} f(x^1, x^2, \dots, x^n)\,\mathrm{d}x^1 .$$

Proof. This formula obviously results from repeated application of the theorem just proved. All the inner integrals on the right-hand side are to be understood as in the theorem. For example, one can insert the upper or lower integral sign throughout. □

Example 1. Let $f(x, y, z) = z \sin(x + y)$. We shall find the integral of the restriction of this function to the interval $I \subset \mathbb{R}^3$ defined by the relations $0 \le x \le \pi$, $|y| \le \pi/2$, $0 \le z \le 1$.

By Corollary 2

$$\iiint_I f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \mathrm{d}z \int_{-\pi/2}^{\pi/2} \mathrm{d}y \int_0^{\pi} z\sin(x+y)\,\mathrm{d}x =$$

$$= \int_0^1 \mathrm{d}z \int_{-\pi/2}^{\pi/2} \left(-z\cos(x+y)\Big|_{x=0}^{\pi}\right)\mathrm{d}y = \int_0^1 \mathrm{d}z \int_{-\pi/2}^{\pi/2} 2z\cos y\,\mathrm{d}y =$$

$$= \int_0^1 \left(2z\sin y\Big|_{y=-\pi/2}^{y=\pi/2}\right)\mathrm{d}z = \int_0^1 4z\,\mathrm{d}z = 2\,.$$

The theorem can also be used to compute integrals over very general sets.

Corollary 3. *Let D be a bounded set in $\mathbb{R}^{n-1}$ and $E = \{(x,y) \in \mathbb{R}^n |\, (x \in D) \wedge (\varphi_1(x) \le y \le \varphi_2(x))\}$. If $f \in \mathcal{R}(E)$, then*

$$\int_E f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_D \mathrm{d}x \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\,. \tag{11.1}$$

Proof. Let $E_x = \{y \in \mathbb{R} |\, \varphi_1(x) \le y \le \varphi_2(x)\}$ if $x \in D$ and $E_x = \varnothing$ if $x \notin D$. We remark that $\chi_E(x,y) = \chi_D(x) \cdot \chi_{E_x}(y)$. Recalling the definition of the integral over a set and using Fubini's theorem, we obtain

$$\int_E f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{I \supset E} f\chi_E(x,y)\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{I_x \supset D} \mathrm{d}x \int_{I_y \supset E_x} f\chi_E(x,y)\,\mathrm{d}y = \int_{I_x} \left(\int_{I_y} f(x,y)\chi_{E_x}(y)\,\mathrm{d}y\right)\chi_D(x)\,\mathrm{d}x =$$

$$= \int_{I_x} \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\right)\chi_D(x)\,\mathrm{d}x = \int_D \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x\,.$$

The inner integral here may also fail to exist on a set of points in D of Lebesgue measure zero, and if so it is assigned the same meaning as in the theorem of Fubini proved above. □

Remark. If the set D in the hypotheses of Corollary 3 is Jordan-measurable and the functions $\varphi_i : D \to \mathbb{R}$, $i = 1, 2$, are continuous, then the set $E \subset \mathbb{R}^n$ is Jordan measurable.

Proof. The boundary ∂E of E consists of the two graphs of the continuous functions $\varphi_i : D \to \mathbb{R}$, $i = 1, 2$, (which by Example 2 of Sect. 11.1) are sets of measure zero) and the set Z, which is a portion of the product of the boundary ∂D of $D \subset \mathbb{R}^{n-1}$ and a sufficiently large one-dimensional closed interval of length l. By hypothesis ∂D can be covered by a system of $(n-1)$-dimensional intervals of total $(n-1)$-dimensional volume less than ε / l. The direct product of these intervals and the given one-dimensional interval of length l gives a covering of Z by intervals whose total volume is less than ε. $\square$

Because of this remark one can say that the function $f : E \to 1 \in \mathbb{R}$ is integrable on a measurable set E having this structure (as it is on any measurable set E). Relying on Corollary 3 and the definition of the measure of a measurable set, one can now derive the following corollary.

Corollary 4. *If under the hypotheses of Corollary 3 the set D is Jordan-measurable and the functions $\varphi_i : D \to \mathbb{R}$, $i = 1, 2$, are continuous, then the set E is measurable and its volume can be computed according to the formula*

$$\mu(E) = \int\limits_D \big(\varphi_2(x) - \varphi_1(x)\big)\, \mathrm{d}x\ . \tag{11.2}$$

Example 2. For the disk $E = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 \le r^2\}$ we obtain by this formula

$$\mu(E) = \int\limits_{-r}^{r} \big(\sqrt{r^2 - y^2} - \big(-\sqrt{r^2 - y^2}\big)\big)\, \mathrm{d}y = 2 \int\limits_{-r}^{r} \sqrt{r^2 - y^2} \mathrm{d}y =$$

$$= 4 \int\limits_0^r \sqrt{r^2 - y^2}\, \mathrm{d}y = 4 \int\limits_0^{\pi/2} r \cos\varphi\, \mathrm{d}(r \sin\varphi) = 4r \int\limits_0^{\pi/2} r \cos^2\varphi\, \mathrm{d}\varphi = \pi r^2\ .$$

Corollary 5. *Let E be a measurable set contained in the interval $I \subset \mathbb{R}^n$. Represent I as the direct product $I = I_x \times I_y$ of the $(n-1)$-dimensional interval I_x and the closed interval I_y. Then for almost all values $y_0 \in I_y$ the section $E_{y_0} = \{(x, y) \in E |\, y = y_0\}$ of the set E by the $(n-1)$-dimensional hyperplane $y = y_0$ is a measurable subset of it, and*

$$\mu(E) = \int\limits_{I_y} \mu(E_y)\, \mathrm{d}y\ , \tag{11.3}$$

where $\mu(E_y)$ is the $(n-1)$-dimensional measure of the set E_y if it is measurable and equal to any number between the numbers $\underline{\int\limits_{E_y}} 1 \cdot \mathrm{d}x$ and $\overline{\int\limits_{E_y}} 1 \cdot \mathrm{d}x$ if E_y happens to be a nonmeasurable set.

Proof. Corollary 5 follows immediately from the theorem and Corollary 1, if we set $f = \chi_E$ in both of them and take account of the relation $\chi_E(x,y) = \chi_{E_y}(x)$. □

A particular consequence of this result is the following.

Corollary 6. (Cavalieri's principle.)[6] *Let A and B be two solids in $\mathbb{R}^3$ having volume (that is, Jordan-measurable). Let $A_c = \{(x,y,z) \in A |\, z = c\}$ and $B_c = \{(x,y,z) \in B |\, z = c\}$ be the sections of the solids A and B by the plane $z = c$. If for every $c \in \mathbb{R}$ the sets A_c and B_c are measurable and have the same area, then the solids A and B have the same volumes.*

It is clear that Cavalieri's principle can be stated for spaces $\mathbb{R}^n$ of any dimension.

Example 3. Using formula (11.3), let us compute the volume V_n of the ball $B = \{x \in \mathbb{R}^n |\, |x| \le r\}$ of radius r in the Euclidean space $\mathbb{R}^n$.

It is obvious that $V_1 = 2$. In Example 2 we found that $V_2 = \pi r^2$. We shall show that $V_n = c_n r^n$, where c_n is a constant (which we shall compute below). Let us choose some diameter $[-r,r]$ of the ball and for each point $x \in [-r,r]$ consider the section B_x of the ball B by a hyperplane orthogonal to the diameter. Since B_x is a ball of dimension $n-1$, whose radius, by the Pythagorean theorem, equals $\sqrt{r^2 - x^2}$, proceeding by induction and using (11.3), we can write

$$V_n = \int_{-r}^{r} c_{n-1}(r^2 - x^2)^{\frac{n-1}{2}}\,\mathrm{d}x = \left(c_{n-1} \int_{-\pi/2}^{\pi/2} \cos^n \varphi\,\mathrm{d}\varphi\right) r^n .$$

(In passing to the last equality, as one can see, we made the change of variable $x = r\sin\varphi$.)

Thus we have shown that $V_n = c_n r^n$, and

$$c_n = c_{n-1} \int_{-\pi/2}^{\pi/2} \cos^n \varphi\,\mathrm{d}\varphi . \tag{11.4}$$

We now find the constant c_n explicitly. We remark that for $m \ge 2$

$$I_m = \int_{-\pi/2}^{\pi/2} \cos^m \varphi\, d\varphi = \int_{-\pi/2}^{\pi/2} \cos^{m-2} \varphi(1 - \sin^2\varphi)\,\mathrm{d}\varphi =$$

$$= I_{m-2} + \frac{1}{m-1} \int_{-\pi/2}^{\pi/2} \sin\varphi\,\mathrm{d}\cos^{m-1}\varphi = I_{m-2} - \frac{1}{m-1} I_m ,$$

[6] B. Cavalieri (1598–1647) – Italian mathematician, the creator of the so-called *method of indivisibles* for determining areas and volumes.

that is, the following recurrence relation holds:

$$I_m = \frac{m-1}{m} I_{m-2} . \tag{11.5}$$

In particular, $I_2 = \pi/2$. It is clear immediately from the definition of I_m that $I_1 = 2$. Taking account of these values of I_1 and I_2 we find by the recurrence formula (11.5) that

$$I_{2k+1} = \frac{(2k)!!}{(2k+1)!!} \cdot 2 , \quad I_{2k} = \frac{(2k-1)!!}{(2k)!!} \pi . \tag{11.6}$$

Returning to formula (11.4), we now obtain

$$c_{2k+1} = c_{2k} \frac{(2k)!!}{(2k+1)!!} \cdot 2 = c_{2k-1} \frac{(2k)!!}{(2k+1)!!} \cdot \frac{(2k-1)!!}{(2k)!!} \pi = \cdots = c_1 \cdot \frac{(2\pi)^k}{(2k+1)!!}$$

$$c_{2k} = c_{2k-1} \frac{(2k-1)!!}{(2k)!!} \pi = c_{2k-2} \frac{(2k-1)!!}{(2k)!!} \pi \cdot \left(\frac{(2k-2)!!}{(2k-1)!!} \cdot 2 = \cdots = c_2 \frac{(2\pi)^{k-1}}{(2k)!!} \cdot 2 .$$

But, as we have seen above, $c_1 = 2$ and $c_2 = \pi$, and hence the final formulas for the required volume V_n are as follows:

$$V_{2k+1} = 2 \frac{(2\pi)^k}{(2k+1)!!} r^{2k+1} , \quad V_{2k} = \frac{(2\pi)^k}{(2k)!!} r^{2k} , \tag{11.7}$$

where $k \in \mathbb{N}$, and the first of these formulas is also valid for $k = 0$.

11.4.3 Problems and Exercises

1. a) Construct a subset of the square $I \subset \mathbb{R}^2$ such that on the one hand its intersection with any vertical line and any horizontal line consists of at most one point, while on the other hand its closure equals I.

b) Construct a function $f : I \to \mathbb{R}$ for which both of the iterated integrals that occur in Fubini's theorem exist and are equal, yet $f \notin \mathcal{R}(I)$.

c) Show by example that if the values of the function $F(x)$ that occurs in Fubini's theorem, which in the theorem were subjected to the conditions $\underline{\mathcal{J}}(x) \leq F(x) \leq \overline{\mathcal{J}}(x)$ at all points where $\underline{\mathcal{J}}(x) < \overline{\mathcal{J}}(x)$, are simply set equal to zero at those points, the resulting function may turn out to be nonintegrable. (Consider, for example, the function $f(x, y)$ on $\mathbb{R}^2$ equal to 1 if the point (x, y) is not rational and to $1 - 1/q$ at the point $(p/q, m/n)$, both fractions being in lowest terms.)

2. a) In connection with formula (11.3), show that even if all the sections of a bounded set E by a family of parallel hyperplanes are measurable, the set E may yet be nonmeasurable.

b) Suppose that in addition to the hypotheses of part a) it is known that the function $\mu(E_y)$ in formula (11.3) is integrable over the closed interval I_y. Can we assert that in this case the set E is measurable?

3. Using Fubini's theorem and the positivity of the integral of a positive function, give a simple proof of the equality $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for the mixed partial derivatives, assuming that they are continuous functions.

4. Let $f : I_{a,b} \to \mathbb{R}$ be a continuous function defined on an interval $I_{a,b} = \{x \in \mathbb{R}^n | a^i \le x^i \le b^i, i = 1, \dots, n\}$, and let $F : I_{a,b} \to \mathbb{R}$ be defined by the equality

$$F(x) = \int\limits_{I_{a,x}} f(t)\,\mathrm{d}t\,,$$

where $I_{a,x} \subset I_{a,b}$. Find the partial derivatives of this function with respect to the variables $x^1, \dots, x^n$.

5. A continuous function $f(x, y)$ defined on the rectangle $I = [a,b] \times [c,d] \subset \mathbb{R}^2$ has a continuous partial derivative $\frac{\partial f}{\partial y}$ in I.

a) Let $F(y) = \int_a^b f(x,y)\,\mathrm{d}x$. Starting from the equality $F(y) = \int_a^b \left(\int_c^y \frac{\partial f}{\partial y}(x,t)\,\mathrm{d}t + f(x,c) \right) \mathrm{d}x$, verify the *Leibniz rule*, according to which $F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$.

b) Let $G(x,y) = \int_a^x f(t,y)\,\mathrm{d}t$. Find $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$.

c) Let $H(y) = \int_a^{h(y)} f(x,y)\,\mathrm{d}x$, where $h \in C^{(1)}[a,b]$. Find $H'(y)$.

6. Consider the sequence of integrals

$$F_0(x) = \int\limits_0^x f(y)\,\mathrm{d}y\,, \quad F_n(x) = \int\limits_0^x \frac{(x-y)^n}{n!} f(y)\,\mathrm{d}y\,, \quad n \in \mathbb{N}\,,$$

where $f \in C(\mathbb{R}, \mathbb{R})$.

a) Verify that $F_n'(x) = F_{n-1}(x)$, $F_n^{(k)}(0) = 0$ if $k \le n$, and $F_n^{(n+1)}(x) = f(x)$.

b) Show that

$$\int\limits_0^x \mathrm{d}x_1 \int\limits_0^{x_1} \mathrm{d}x_2 \cdots \int\limits_0^{x_{n-1}} f(x_n)\,\mathrm{d}x_n = \frac{1}{n!} \int\limits_0^x (x-y)^n f(y)\,\mathrm{d}y\,.$$

7. a) Let $f : E \to \mathbb{R}$ be a function that is continuous on the set $E = \{(x,y) \in \mathbb{R}^2 | 0 \le x \le 1 \wedge 0 \le y \le x\}$. Prove that

$$\int\limits_0^1 \mathrm{d}x \int\limits_0^x f(x,y)\,\mathrm{d}y = \int\limits_0^1 \mathrm{d}y \int\limits_y^1 f(x,y)\,\mathrm{d}x\,.$$

b) Use the example of the iterated integral $\int_0^{2\pi} \mathrm{d}x \int_0^{\sin x} 1 \cdot \mathrm{d}y$ to explain why not every iterated integral comes from a double integral via Fubini's theorem.

11.5 Change of Variable in a Multiple Integral

11.5.1 Statement of the Problem and Heuristic Derivation of the Change of Variable Formula

In our earlier study of the integral in the one-dimensional case, we obtained an important formula for change of variable in such an integral. Our problem now is to find a formula for change of variables in the general case. Let us make the question more precise.

Let D_x be a set in $\mathbb{R}^n$, f a function that is integrable over D_x, and $\varphi : D_t \to D_x$ a mapping $t \mapsto \varphi(t)$ of a set $D_t \subset \mathbb{R}^n$ onto D_x. We seek a rule according to which, knowing f and φ, we can find a function ψ in D_t such that the equality

$$\int_{D_x} f(x)\,\mathrm{d}x = \int_{D_t} \psi(t)\,\mathrm{d}t$$

holds, making it possible to reduce the computation of the integral over D_x to the computation of an integral over D_t.

We begin by assuming that D_t is an interval $I \subset \mathbb{R}^n$ and $\varphi : I \to D_x$ a diffeomorphism of this interval onto D_x. To every partition P of the interval I into intervals $I_1, I_2, \dots, I_k$ there corresponds a partition of D_x into the sets $\varphi(I_i)$, $i = 1, \dots, k$. If all these sets are measurable and intersect pairwise only in sets of measure zero, then by the additivity of the integral we find

$$\int_{D_x} f(x)\,\mathrm{d}x = \sum_{i=1}^{k} \int_{\varphi(I_i)} f(x)\,\mathrm{d}x\,. \tag{11.8}$$

If f is continuous on D_x, then by the mean-value theorem

$$\int_{\varphi(I_i)} f(x)\,\mathrm{d}x = f(\xi_i)\mu\big(\varphi(I_i)\big)\,,$$

where $\xi_i \in \varphi(I_i)$. Since $f(\xi_i) = f\big(\varphi(\tau_i)\big)$, where $\tau_i = \varphi^{-1}(\xi_i)$, we need only connect $\mu\big(\varphi(I_i)\big)$ with $\mu(I_i)$.

If φ were a linear transformation, then $\varphi(I_i)$ would be a parallelepiped whose volume, as is known from analytic geometry, would be $|\det \varphi'|\mu(I_i)$. But a diffeomorphism is locally a nearly linear transformation, and so, if the dimensions of the intervals I_i are sufficiently small, we may assume $\mu\big(\varphi(I_i)\big) \approx |\det \varphi'(\tau_i)|\,|I_i|$ with small relative error (it can be shown that for some choice of the point $\tau_i \in I_i$ actual equality will result). Thus

$$\sum_{i=1}^{k} \int_{\varphi(I_i)} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{k} f\big(\varphi(\tau_i)\big)|\det \varphi'(\tau_i)|\,|I_i|\,. \tag{11.9}$$

But, the right-hand side of this approximate equality contains a Riemann sum for the integral of the function $f(\varphi(t))|\det \varphi'(t)|$ over the interval I corresponding to the partition P of this interval with distinguished points τ. In the limit as $\lambda(P) \to 0$ we obtain from (11.8) and (11.9) the relation

$$\int_{D_x} f(x)\,\mathrm{d}x = \int_{D_t} f(\varphi(t))|\det \varphi'(t)|\,\mathrm{d}t\,.$$

This is the desired formula together with an explanation of it. The route just followed in obtaining it can be traversed with complete rigor (and it is worthwhile to do so). However, in order to become acquainted with some new and useful general mathematical methods and facts and avoid purely technical work, we shall depart from this route slightly in the proof below.

We now proceed to precise statements. We recall the following definition.

Definition 1. The *support of a function* $f : D \to \mathbb{R}$ defined in a domain $D \subset \mathbb{R}^n$ is the closure in D of the set of points of $x \in D$ at which $f(x) \neq 0$.

In this section we shall study the situation when the integrand $f : D_x \to \mathbb{R}$ equals zero on the boundary of the domain D_x, more precisely, when the support of the function f (denoted $\operatorname{supp} f$) is a compact set[7] K contained in D_x. The integrals of f over D_x and over K, if they exist, are equal, since the function equals zero in D_x outside of K. From the point of view of mappings the condition $\operatorname{supp} f = K \subset D_x$ is equivalent to the statement that the change of variable $x = \varphi(t)$ is valid not only in the set K over which one is essentially integrating, but also in some neighborhood D_x of that set.

We now state what we intend to prove.

Theorem 1. *If $\varphi : D_t \to D_x$ is a diffeomorphism of a bounded open set $D_t \subset \mathbb{R}^n$ onto a set $D_x = \varphi(D_t) \subset \mathbb{R}^n$ of the same type, $f \in \mathcal{R}(D_x)$, and* $\operatorname{supp} f$ *is a compact subset of D_x, then $f \circ \varphi|\det \varphi'| \in \mathcal{R}(D_t)$, and the following formula holds:*

$$\boxed{\int_{D_x=\varphi(D_t)} f(x)\,\mathrm{d}x = \int_{D_t} f\circ\varphi(t)|\det \varphi'(t)|\,\mathrm{d}t\,.} \tag{11.10}$$

11.5.2 Measurable Sets and Smooth Mappings

Lemma 1. *Let $\varphi : D_t \to D_x$ be a diffeomorphism of an open set $D_t \subset \mathbb{R}^n$ onto a set $D_x \subset \mathbb{R}^n$ of the same type. Then the following assertions hold.*

a) If $E_t \subset D_t$ is a set of (Lebesgue) measure zero, its image $\varphi(E_t) \subset D_x$ is also a set of measure zero.

[7] Such functions are naturally called *functions of compact support* in the domain.

b) If a set E_t contained in D_t along with its closure $\overline{E}_t$ has Jordan measure zero, its image $\varphi(E_t) = E_x$ is contained in D_x along with its closure and also has measure zero.

c) If a (Jordan) measurable set E_t is contained in the domain D_t along with its closure $\overline{E}_t$, its image $E_x = \varphi(E_t)$ is Jordan measurable and $\overline{E}_x \subset D_x$.

Proof. We begin by remarking that every open subset D in $\mathbb{R}^n$ can be represented as the union of a countable number of closed intervals (no two of which have any interior points in common). To do this, for example, one can partition the coordinate axes into closed intervals of length Δ and consider the corresponding partition of $\mathbb{R}^n$ into cubes with sides of length Δ. Fixing $\Delta = 1$, take the cubes of the partition contained in D. Denote their union by F_1. Then taking $\Delta = 1/2$, adjoin to F_1 the cubes of the new partition that are contained in $D \setminus F_1$. In that way we obtain a new set F_2, and so forth. Continuing this process, we obtain a sequence $F_1 \subset \cdots \subset F_n \subset \cdots$ of sets, each of which consists of a finite or countable number of intervals having no interior points in common, and as one can see from the construction, $\bigcup F_n = D$.

Since the union of an at most countable collection of sets of measure zero is a set of measure zero, it suffices to verify assertion $a)$ for a set E_t lying in a closed interval $I \subset D_t$. We shall now do this.

Since $\varphi \in C^{(1)}(I)$ (that is, $\varphi' \in C(I)$), there exists a constant M such that $\|\varphi'(t)\| \leq M$ on I. By the finite-increment theorem the relation $|x_2 - x_1| \leq M|t_2 - t_1|$ must hold for every pair of points $t_1, t_2 \in I$ with images $x_1 = \varphi(t_1)$, $x_2 = \varphi(t_2)$.

Now let $\{I_i\}$ be a covering of E_t by intervals such that $\sum\limits_i |I_i| < \varepsilon$. Without loss of generality we may assume that $I_i = I_i \cap I \subset I$.

The collection $\{\varphi(I_i)\}$ of sets $\varphi(I_i)$ obviously forms a covering of $E_x = \varphi(E_t)$. If t_i is the center of the interval I_i, then by the estimate just given for the possible change in distances under the mapping φ, the entire set $\varphi(I_i)$ can be covered by the interval $\widetilde{I}_i$ with center $x_i = \varphi(t_i)$ whose linear dimensions are M times those of the interval I_i. Since $|\widetilde{I}_i| = M^n|I_i|$, and $\varphi(E_t) \subset \bigcup\limits_i \widetilde{I}_i$, we have obtained a covering of $\varphi(E_t) = E_x$ by intervals whose total volume is less than $M^n\varepsilon$. Assertion $a)$ is now established.

Assertion $b)$ follows from $a)$ if we take into account the fact that $\overline{E}_t$ (and hence by what has been proved, $\overline{E}_x = \varphi(\overline{E}_t)$ also) is a set of Lebesgue measure zero and that $\overline{E}_t$ (and hence also $\overline{E}_x$) is a compact set. Indeed, by Lemma 3 of Sect. 11.1 every compact set that is of Lebesgue measure zero also has Jordan measure zero.

Finally, assertion $c)$ is an immediate consequence of $b)$, if we recall the definition of a measurable set and the fact that interior points of E_t map to interior points of its image $E_x = \varphi(E_t)$ under a diffeomorphism, so that $\partial E_x = \varphi(\partial E_t)$. $\square$

Corollary. *Under the hypotheses of the theorem the integral on the right-hand side of formula* (11.10) *exists.*

Proof. Since $|\det \varphi'(t)| \neq 0$ in D_t, it follows that $\operatorname{supp} f \circ \varphi \cdot |\det \varphi'| = \operatorname{supp} f \circ \varphi \circ \varphi^{-1}(\operatorname{supp} f)$ is a compact subset in D_t. Hence the points at which the function $f \circ \varphi \cdot |\det \varphi'| \chi_{D_t}$ in $\mathbb{R}^n$ is discontinuous have nothing to do with the function χ_{D_t}, but are the pre-images of points of discontinuity of f in D_x. But $f \in \mathcal{R}(D_x)$, and therefore the set E_x of points of discontinuity of f in D_x is a set of Lebesgue measure zero. But then by assertion $a)$ of the lemma the set $E_t = \varphi^{-1}(E_x)$ has measure zero. By Lebesgue's criterion, we can now conclude that $f \circ \varphi \cdot |\det \varphi'| \chi_{D_t}$ is integrable on any interval $I_t \supset D_t$. □

11.5.3 The One-dimensional Case

Lemma 2. *a) If $\varphi : I_t \to I_x$ is a diffeomorphism of a closed interval $I_t \subset \mathbb{R}^1$ onto a closed interval $I_x \subset \mathbb{R}^1$ and $f \in \mathcal{R}(I_x)$, then $f \circ \varphi \cdot |\varphi'| \in \mathcal{R}(I_t)$ and*

$$\int_{I_x} f(x)\,\mathrm{d}x = \int_{I_t} (f \circ \varphi \cdot |\varphi'|)(t)\,\mathrm{d}t\,. \tag{11.11}$$

b) Formula (11.10) *holds in* $\mathbb{R}^1$.

Proof. Although we essentially already know assertion $a)$ of this lemma, we shall use the Lebesgue criterion for the existence of an integral, which is now at our disposal, to give a short proof here that is independent of the proof given in Part 1.

Since $f \in \mathcal{R}(I_x)$ and $\varphi : I_t \to I_x$ is a diffeomorphism, the function $f \circ \varphi |\varphi'|$ is bounded on I_t. Only the preimages of points of discontinuity of f on I_x can be discontinuities of the function $f \circ \varphi |\varphi'|$. By Lebesgue's criterion, the latter form a set of measure zero. The image of this set under the diffeomorphism $\varphi^{-1} : I_x \to I_t$, as we saw in the proof of Lemma 1, has measure zero. Therefore $f \circ \varphi |\varphi'| \in \mathcal{R}(I_t)$.

Now let P_x be a partition of the closed interval I_x. Through the mapping φ^{-1} it induces a partition P_t of the closed interval I_t, and it follows from the uniform continuity of the mappings φ and φ^{-1} that $\lambda(P_x) \to 0 \Leftrightarrow \lambda(P_t) \to 0$. We now write the Riemann sums for the partitions P_x and P_t with distinguished points $\xi_i = \varphi(\tau_i)$:

$$\sum_i f(\xi_i)|x_i - x_{i-1}| = \sum_i f \circ \varphi(\tau_i)|\varphi(t_i) - \varphi(t_{i-1}| =$$
$$= \sum_i f \circ \varphi(\tau_i)|\varphi'(\tau_i)|\,|t_i - t_{i-1}|\,,$$

and the points ξ_i can be assumed chosen just so that $\xi_i = \varphi(\tau_i)$, where τ_i is the point obtained by applying the mean-value theorem to the difference $\varphi(t_i) - \varphi(t_{i-1})$.

Since both integrals in (11.11) exist, the choice of the distinguished points in the Riemann sums can be made to suit our convenience without affecting the limit. Hence from the equalities just written for the Riemann sums, we find (11.11) for the integrals in the limit as $\lambda(P_x) \to 0$ ($\lambda(P_t) \to 0$).

Assertion b) of Lemma 2 follows from Eq. (11.11). We first note that in the one-dimensional case $|\det \varphi'| = |\varphi'|$. Next, the compact set $\operatorname{supp} f$ can easily be covered by a finite system of closed intervals contained in D_x, no two of which have common interior points. The integral of f over D_x then reduces to the sum of the integrals of f over the intervals of this system, and the integral of $f \circ \varphi|\varphi'|$ over D_t reduces to the sum of the integrals over the intervals that are the pre-images of the intervals in this system. Applying Eq. (11.11) to each pair of intervals that correspond under the mapping φ and then adding, we obtain (11.10). □

Remark 1. The formula for change of variable that we proved previously had the form

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,\mathrm{d}x = \int_{\alpha}^{\beta} \big((f \circ \varphi) \cdot \varphi'\big)(t)\,\mathrm{d}t\,, \tag{11.12}$$

where φ was any smooth mapping of the closed interval $[\alpha, \beta]$ onto the interval with endpoints $\varphi(\alpha)$ and $\varphi(\beta)$. Formula (11.12) contains the derivative φ' itself rather than its absolute value $|\varphi'|$. The reason is that on the left-hand side it is possible that $\varphi(\beta) < \varphi(\alpha)$.

However, if we observe that the relations

$$\int_I f(x)\,\mathrm{d}x = \begin{cases} \int_a^b f(x)\,\mathrm{d}x, & \text{if} \quad a \le b\,, \\ -\int_a^b f(x)\,\mathrm{d}x, & \text{if} \quad a > b\,, \end{cases}$$

hold, it becomes clear that when φ is a diffeomorphism formulas (11.11) and (11.12) differ only in appearance; in essence they are the same.

Remark 2. It is interesting to note (and we shall certainly make use of this observation) that if $\varphi : I_t \to I_x$ is a diffeomorphism of closed intervals, then the formulas

$$\overline{\int_{I_x}} f(x)\,\mathrm{d}x = \overline{\int_{I_t}} \big(f \circ \varphi|\varphi'|\big)(t)\,\mathrm{d}t\,,$$

$$\underline{\int_{I_x}} f(x)\,\mathrm{d}x = \underline{\int_{I_t}} \big(f \circ \varphi|\varphi'|\big)(t)\,\mathrm{d}t\,,$$

for the upper and lower integrals of real-valued functions are always valid.

Given that fact, we may take as established that in the one-dimensional case formula (11.10) remains valid for any bounded function f if the integrals in it are understood as upper or lower Darboux integrals.

Proof. We shall assume temporarily that f is a nonnegative function bounded by a constant M.

Again, as in the proof of assertion $a)$ of Lemma 2, one may take partitions P_x and P_t of the intervals I_x and I_t respectively that correspond to each other under the mapping φ and write the following estimates, in which ε is the maximum oscillation of φ on intervals of the partition P_t:

$$\sum_i \sup_{x\in\Delta x_i} f(x)|x_i - x_{i-1}| \le \sum_i \sup_{t\in\Delta t_i} f\big(\varphi(t)\big) \sup_{t\in\Delta t_i} \big|\varphi'(t)\big|\,|t_i - t_{i-1}| \le$$
$$\le \sum_i \sup_{t\in\Delta t_i} \Big(f\big(\varphi(t)\big)\cdot \sup_{t\in\Delta t_i} |\varphi'(t)|\Big)|\Delta t_i| \le$$
$$\le \sum_i \sup_{t\in\Delta t_i} \big(f(\varphi(t))\big)\big(|\varphi'(t)| + \varepsilon\big)|\Delta t_i| \le$$
$$\le \sum_i \sup_{t\in\Delta t_i} \big(f(\varphi(t))|\varphi'(t)|\big)|\Delta t_i| + \varepsilon \sum_i \sup_{t\in\Delta t_i} f\big(\varphi(t)\big)|\Delta t_i| \le$$
$$\le \sum_i \sup_{t\in\Delta t_i} \big(f(\varphi(t))|\varphi'(t)|\big)|\Delta t_i| + \varepsilon M|I_t|\,.$$

Taking account of the uniform continuity of φ we obtain from this the relation

$$\overline{\int_{I_x}} f(x)\,\mathrm{d}x \le \overline{\int_{I_t}} \big(f\circ\varphi|\varphi'|\big)(t)\,\mathrm{d}t$$

as $\lambda(P_t) \to 0$. Applying what has just been proved to the mapping φ^{-1} and the function $f\circ\varphi|\varphi'|$, we obtain the opposite inequality, and thereby establish the first equality in Remark 2 for a nonnegative function. But since any function can be written as $f = \max\{f, 0\} - \max\{-f, 0\}$ (a difference of two nonnegative functions) the equality can be considered to be established in general. The second equality is verified similarly. □

From the equalities just proved one can of course obtain once again assertion $a)$ of Lemma 2 for real-valued functions f.

11.5.4 The Case of an Elementary Diffeomorphism in $\mathbb{R}^n$

Let $\varphi: D_t \to D_x$ be a diffeomorphism of a domain $D_t \subset \mathbb{R}^n_t$ onto a domain $D_x \subset \mathbb{R}^n_x$ with $(t^1, \dots, t^n)$ and $(x^1, \dots, x^n)$ the coordinates of points $t \in \mathbb{R}^n_t$ and $x \in \mathbb{R}^n_x$ respectively. We recall the following definition.

Definition 2. The diffeomorphism $\varphi: D_t \to D_x$ is *elementary* if its coordinate representation has the form

$$x^1 = \varphi^1(t^1, \dots, t^n) = t^1 \ ,$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$x^{k-1} = \varphi^{k-1}(t^1, \dots, t^n) = t^{k-1} \ ,$$
$$x^k = \varphi^k(t^1, \dots, t^n) = \varphi^k(t^1, \dots, t^k, \dots, t^n) \ ,$$
$$x^{k+1} = \varphi^k(t^1, \dots, t^n) = t^{k+1}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$x^n = \varphi^n(t^1, \dots, t^n) = t^n \ .$$

Thus only one coordinate is changed under an elementary diffeomorphism (the kth coordinate in this case).

Lemma 3. *Formula* (11.10) *holds for an elementary diffeomorphism.*

Proof. Up to a relabeling of coordinates we may assume that we are considering a diffeomorphism φ that changes only the nth coordinate. For convenience we introduce the following notation:

$$(x^1, \dots, x^{n-1}, x^n) =: (\widetilde{x}, x^n) \ ; \quad (t^1, \dots, t^{n-1}, t^n) =: (\widetilde{t}, t^n) \ ;$$
$$D_{x^n}(\widetilde{x}_0) := \{(\widetilde{x}, x^n) \in D_x | \, \widetilde{x} = \widetilde{x}_0\} \ ;$$
$$D_{t_n}(\widetilde{t}_0) := \{(\widetilde{t}, t^n) \in D_t | \, \widetilde{t} = \widetilde{t}_0\} \ .$$

Thus $D_{x^n}(\widetilde{x})$ and $D_{t^n}(\widetilde{t})$ are simply the one-dimensional sections of the sets D_x and D_t respectively by lines parallel to the nth coordinate axis. Let I_x be an interval in $\mathbb{R}^n_x$ containing D_x. We represent I_x as the direct product $I_x = I_{\widetilde{x}} \times I_{x^n}$ of an $(n-1)$-dimensional interval $I_{\widetilde{x}}$ and a closed interval I_{x^n} of the nth coordinate axis. We give a similar representation $I_t = I_{\widetilde{t}} \times I_{t^n}$ for a fixed interval I_t in $\mathbb{R}^n_t$ containing D_t.

Using the definition of the integral over a set, Fubini's theorem, and Remark 2, we can write

$$\int\limits_{D_x} f(x)\,\mathrm{d}x = \int\limits_{I_x} f \cdot \chi_{D_x}(x)\,\mathrm{d}x = \int\limits_{I_{\widetilde{x}}} \mathrm{d}\widetilde{x} \int\limits_{I_{x^n}} f \cdot \chi_{D_x}(\widetilde{x}, x^n)\,\mathrm{d}x^n =$$
$$= \int\limits_{I_{\widetilde{x}}} \mathrm{d}\widetilde{x} \int\limits_{D_{x^n}(\widetilde{x})} f(\widetilde{x}, x^n)\,\mathrm{d}x^n =$$
$$= \int\limits_{I_{\widetilde{t}}} \mathrm{d}\widetilde{t} \int\limits_{D_{t^n}(\widetilde{t})} f\big(\widetilde{t}, \varphi^n(\widetilde{t}, t^n)\big) \left|\frac{\partial \varphi^n}{\partial t^n}\right| (\widetilde{t}, t^n)\,\mathrm{d}t^n =$$
$$= \int\limits_{I_{\widetilde{t}}} \mathrm{d}\widetilde{t} \int\limits_{I_{t^n}} (f \circ \varphi | \det \varphi' | \chi_{D_t})(\widetilde{t}, t^n)\,\mathrm{d}t^n =$$
$$= \int\limits_{I_t} (f \circ \varphi | \det \varphi' | \chi_{D_t})(t)\,\mathrm{d}t = \int\limits_{D_t} (f \circ \varphi | \det \varphi' |)(t)\,\mathrm{d}t \ .$$

In this computation we have used the fact that $\det \varphi' = \frac{\partial \varphi^n}{\partial t^n}$ for the diffeomorphism under considertation. □

11.5.5 Composite Mappings and the Formula for Change of Variable

Lemma 4. *If $D_\tau \xrightarrow{\psi} D_t \xrightarrow{\varphi} D_x$ are two diffeomorphisms for each of which formula* (11.10) *for change of variable in the integral holds, then it holds also for the composition $\varphi \circ \psi : D_\tau \to D_x$ of these mappings.*

Proof. It suffices to recall that $(\varphi \circ \psi)' = \varphi' \circ \psi'$ and that $\det(\varphi \circ \psi)'(\tau) = \det \varphi'(t) \det \psi'(\tau)$, where $t = \varphi(\tau)$. We then have

$$\int\limits_{D_x} f(x)\,\mathrm{d}x = \int\limits_{D_t} \big(f \circ \varphi |\det \varphi'|\big)\,\mathrm{d}t =$$

$$= \int\limits_{D_\tau} \big((f \circ \varphi \circ \psi)|\det \varphi' \circ \psi|\,|\det \psi'|\big)(\tau)\,\mathrm{d}\tau =$$

$$= \int\limits_{D_\tau} \big(f \circ (\varphi \circ \psi)|\det(\varphi \circ \psi)'|\big)(\tau)\,\mathrm{d}\tau\,. \ \square$$

11.5.6 Additivity of the Integral and Completion of the Proof of the Formula for Change of Variable in an Integral

Lemmas 3 and 4 suggest that we might use the local decomposition of any diffeomorphism as a composition of elementary diffeomorphisms (see Proposition 2 from Subsect. 8.6.4 of Part 1) and thereby obtain the formula (11.10) in the general case.

There are various ways of reducing the integral over a set to integrals over small neighborhoods of its points. For example, one may use the additivity of the integral. That is the procedure we shall use. On the basis of Lemmas 1, 3, and 4 we now carry out the proof of Theorem 1 on change of variable in a multiple integral.

Proof. For each point t of the compact set $K_t = \operatorname{supp}\big((f \circ \varphi)|\det \varphi'|\big) \subset D_t$ we construct a $\delta(t)$-neighborhood $U(t)$ of it in which the diffeomorphism φ decomposes into a composition of elementary diffeomorphisms. From the $\frac{\delta(t)}{2}$-neighborhoods $\widetilde{U}(t) \subset U(T)$ of the points $t \in K_t$ we choose a finite covering $\widetilde{U}(t_1), \dots, \widetilde{U}(t_k)$ of the compact set K_t. Let $\delta = \frac{1}{2}\min\{\delta(t_1), \dots, \delta(t_k)\}$. Then the closure of any set whose diameter is smaller than δ and which intersects K_t must be contained in at least one of the neighborhoods $\widetilde{U}(t_1), \dots, \widetilde{U}(t_k)$.

Now let I be an interval containing the set D_t and P a partition of the interval I such that $\lambda(P) < \min\{\delta, d\}$, where δ was found above and d is the distance from K_t to the boundary of D_t. Let $\{I_i\}$ be the intervals of the partition P that have a nonempty intersection with K_t. It is clear that if $I_i \in \{I_i\}$, then $I_i \subset D_t$ and

$$\int\limits_{D_t} (f \circ \varphi |\det \varphi'|)(t)\,dt = \int\limits_{I} ((f \circ \varphi |\det \varphi'|)\chi_{D_t})(t)\,dt = \\ = \sum_i \int\limits_{I_i} (f \circ \varphi |\det \varphi'|)(t)\,dt\,. \qquad (11.13)$$

By Lemma 1 the image $E_i = \varphi(I_i)$ of the intervals I_i is a measurable set. Then the set $E = \bigcup\limits_i E_i$ is also measurable and $\operatorname{supp} f \subset E = \overline{E} \subset D_x$. Using the additivity of the integral, we deduce from this that

$$\int\limits_{D_x} f(x)\,dx = \int\limits_{I_x \subset D_x} f\chi_{D_x}(x)\,dx = \int\limits_{I_x \setminus E} f\chi_{D_x}(x)\,dx + \int\limits_{E} f\chi_{D_x}(x)\,dx = \\ = \int\limits_{E} f\chi_{D_x}(x)\,dx = \int\limits_{E} f(x)\,dx = \sum_i \int\limits_{E_i} f(x)\,dx\,. \qquad (11.14)$$

By construction every interval $I_i \in \{I_i\}$ is contained in some neighborhood $U(x_j)$ inside which the diffeomorphism φ decomposes into a composition of elementary diffeomorphisms. Hence on the basis of Lemmas 3 and 4 we can write

$$\int\limits_{E_i} f(x)\,dx = \int\limits_{I_i} (f \circ \varphi |\det \varphi'|)(t)\,dt\,. \qquad (11.15)$$

Comparing relations (11.13), (11.14), and (11.15), we obtain formula (11.10). □

11.5.7 Corollaries and Generalizations of the Formula for Change of Variable in a Multiple Integral

a. Change of Variable under Mappings of Measurable Sets

Proposition 1. *Let $\varphi : D_t \to D_x$ be a diffeomorphism of a bounded open set $D_t \subset \mathbb{R}^n$ onto a set $D_x \subset \mathbb{R}^n$ of the same type; let E_t and E_x be subsets of D_t and D_x respectively and such that $\overline{E}_t \subset D_t$, $\overline{E}_x \subset D_x$, and $E_x = \varphi(E_t)$. If $f \in \mathcal{R}(E_x)$, then $f \circ \varphi |\det \varphi'| \in \mathcal{R}(E_t)$, and the following equality holds:*

$$\int\limits_{E_x} f(x)\,dx = \int\limits_{E_t} (f \circ \varphi |\det \varphi'|)(t)\,dt\,. \qquad (11.16)$$

Proof. Indeed,

$$\int_{E_x} f(x)\,\mathrm{d}x = \int_{D_x} (f\chi_{E_x})(x)\,\mathrm{d}x = \int_{D_t} \big(((f\chi_{E_x})\circ\varphi)|\det\varphi'|\big)(t)\,\mathrm{d}t =$$
$$= \int_{D_t} \big((f\circ\varphi)|\det\varphi'|\chi_{E_t}\big)(t)\,\mathrm{d}t = \int_{E_t} \big((f\circ\varphi)|\det\varphi'|\big)(t)\,\mathrm{d}t\,.$$

In this computation we have used the definition of the integral over a set, formula (11.10), and the fact that $\chi_{E_t} = \chi_{E_x}\circ\varphi$. □

b. Invariance of the Integral We recall that the integral of a function $f : E \to \mathbb{R}$ over a set E reduces to computing the integral of the function $f\chi_E$ over an interval $I \supset E$. But the interval I itself was by definition connected with a Cartesian coordinate system in $\mathbb{R}^n$. We can now prove that all Cartesian systems lead to the same integral.

Proposition 2. *The value of the integral of a function f over a set $E \subset \mathbb{R}^n$ is independent of the choice of Cartesian coordinate system in $\mathbb{R}^n$.*

Proof. In fact the transition from one Cartesian coordinate system in $\mathbb{R}^n$ to another Cartesian system has a Jacobian constantly equal to 1 in absolute value. By Proposition 1 this implies the equality

$$\int_{E_x} f(x)\,\mathrm{d}x = \int_{E_t} (f\circ\varphi)(t)\,\mathrm{d}t\,.$$

But this means that the integral is invariantly defined: if p is a point of E having coordinates $x = (x^1,\dots,x^n)$ in the first system and $t = (t^1,\dots,t^n)$ in the second, and $x = \varphi(t)$ is the transition function from one system to the other, then

$$f(p) = f_x(x^1,\dots,x^n) = f_t(t^1,\dots,t^n)\,,$$

where $f_t = f_x\circ\varphi$. Hence we have shown that

$$\int_{E_x} f_x(x)\,\mathrm{d}x = \int_{E_t} f_t(t)\,\mathrm{d}t\,,$$

where E_x and E_t denote the set E in the x and t coordinates respectively. □

We can conclude from Proposition 2 and Definition 3 of Sect. 11.2 for the (Jordan) measure of a set $E \subset \mathbb{R}^n$ that this measure is independent of the Cartesian coordinate system in $\mathbb{R}^n$, or, what is the same, that Jordan measure is invariant under the group of rigid Euclidean motions in $\mathbb{R}^n$.

c. Negligible Sets The changes of variable or formulas for transforming coordinates used in practice sometimes have various singularities (for example, one-to-oneness may fail in some places, or the Jacobian may vanish, or differentiability may fail). As a rule, these singularities occur on a set of measure zero and so, to meet the demands of practice, the following theorem is very useful.

Theorem 2. *Let $\varphi : D_t \to D_x$ be a mapping of a (Jordan) measurable set $D_t \subset \mathbb{R}^n_t$ onto a set $D_x \subset \mathbb{R}^n_x$ of the same type. Suppose that there are subsets S_t and S_x of D_t and D_x respectively having (Lebesgue) measure zero and such that $D_t \setminus S_t$ and $D_x \setminus S_x$ are open sets and φ maps the former diffeomorphically onto the latter and with a bounded Jacobian. Then for any function $f \in \mathcal{R}(D_x)$ the function $(f \circ \varphi)|\det \varphi'|$ also belongs to $\mathcal{R}(D_t \setminus S_t)$ and*

$$\int\limits_{D_x} f(x)\,\mathrm{d}x = \int\limits_{D_t \setminus S_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t\,. \tag{11.17}$$

If, in addition, the quantity $|\det \varphi'|$ is defined and bounded in D_t, then

$$\int\limits_{D_x} f(x)\,\mathrm{d}x = \int\limits_{D_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t\,. \tag{11.18}$$

Proof. By Lebesgue's criterion the function f can have discontinuities in D_x and hence also in $D_x \setminus S_x$ only on a set of measure zero. By Lemma 1, the image of this set of discontinuities under the mapping $\varphi^{-1} : D_x \setminus S_x \to D_t \setminus S_t$ is a set of measure zero in $D_t \setminus S_t$. Thus the relation $(f \circ \varphi)|\det \varphi'| \in \mathcal{R}(D_t \setminus S_t)$ will follow immediately from Lebesgue's criterion for integrability if we establish that the set $D_t \setminus S_t$ is measurable. The fact that this is indeed a Jordan measurable set will be a by-product of the reasoning below.

By hypothesis $D_x \setminus S_x$ is an open set, so that $(D_x \setminus S_x) \cap \partial S_x = \varnothing$. Hence $\partial S_x \subset \partial D_x \cup S_x$ and consequently $\partial D_x \cup S_x = \partial D_x \cup \overline{S}_x$, where $\overline{S}_x = S_x \cup \partial S_x$ is the closure of S_x in $\mathbb{R}^n_x$. As a result, $\partial D_x \cup S_x$ is a closed bounded set, that is, it is compact in $\mathbb{R}^n$, and, being the union of two sets of measure zero, is itself of Lebesgue measure zero. From Lemma 3 of Sect. 11.1 we know that then the set $\partial D_x \cup S_x$ (and along with it, S_x) has measure zero, that is, for every $\varepsilon > 0$ there exists a finite covering $I_1, \dots, I_k$ of this set by intervals such that $\sum\limits_{i=1}^{k} |I_i| < \varepsilon$. Hence it follows, in particular, that the set $D_x \setminus S_x$ (and similarly the set $D_t \setminus S_t$) is Jordan measurable: indeed, $\partial(D_x \setminus S_x) \subset \partial D_x \cup \partial S_x \subset \partial D_x \cup S_x$.

The covering $I_1, \dots, I_k$ can obviously also be chosen so that every point $x \in \partial D_x \setminus S_x$ is an interior point of at least one of the intervals of the covering. Let $U_x = \bigcup_{i=1}^{k} I_i$. The set U_x is measurable, as is $V_x = D_x \setminus U_x$. By construction the set V_x is such that $\overline{V}_x \subset D_x \setminus S_x$ and for every measurable set $E_x \subset D_x$ containing the compact set $\overline{V}_x$ we have the estimate

$$\left| \int\limits_{D_x} f(x)\,\mathrm{d}x - \int\limits_{E_x} f(x)\mathrm{d}x \right| = \left| \int\limits_{D_x \setminus E_x} f(x)\,\mathrm{d}x \right| \le$$
$$\le M\mu(D_x \setminus E_x) < M \cdot \varepsilon \,, \qquad (11.19)$$

where $M = \sup\limits_{x \in D_x} f(x)$.

The pre-image $\overline{V}_t = \varphi^{-1}(\overline{V}_x)$ of the compact set $\overline{V}_x$ is a compact subset of $D_t \setminus S_t$. Reasoning as above, we can construct a measurable compact set W_t subject to the conditions $\overline{V}_t \subset W_t \subset D_t \setminus S_t$ and having the property that the estimate

$$\left| \int\limits_{D_t \setminus S_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t - \int\limits_{E_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t \right| < \varepsilon \qquad (11.20)$$

holds for every measurable set E_t such that $W_t \subset E_t \subset D_t \setminus S_t$.

Now let $E_x = \varphi(E_t)$. Formula (11.16) holds for the sets $E_x \subset D_x \setminus S_x$ and $E_t \subset D_t \setminus S_t$ by Lemma 1. Comparing relations (11.16), (11.19), and (11.20) and taking account of the arbitrariness of the quantity $\varepsilon > 0$, we obtain (11.17).

We now prove the last assertion of Theorem 2. If the function $(f \circ \varphi)|\det \varphi'|$ is defined on the entire set D_t, then, since $D_t \setminus S_t$ is open in $\mathbb{R}^n_t$, the entire set of discontinuities of this function in D_t consists of the set A of points of discontinuity of $(f \circ \varphi)|\det \varphi'|\big|_{D_t \setminus S_t}$ (the restriction of the original function to $D_t \setminus S_t$) and perhaps a subset B of $S_t \cup \partial D_t$.

As we have seen, the set A is a set of Lebesgue measure zero (since the integral on the right-hand side of (11.17) exists), and since $S_t \cup \partial D_t$ has measure zero, the same can be said of B. Hence it suffices to know that the function $(f \circ \varphi)|\det \varphi'|$ is bounded on D_t; it will then follow from the Lebesgue criterion that it is integrable over D_t. But $|f \circ \varphi|(t) \le M$ on D_t, so that the function $(f \circ \varphi)|\det \varphi'|$ is bounded on S_t, given that the function $|\det \varphi'|$ is bounded on S_t by hypothesis. As for the set $D_t \setminus S_t$, the function $(f \circ \varphi)|\det \varphi'|$ is integrable over it and hence bounded. Thus, the function $(f \circ \varphi)|\det \varphi'|$ is integrable over D_t. But the sets D_t and $D_t \setminus S_t$ differ only by the measurable set S_t, whose measure, as has been shown, is zero. Therefore, by the additivity of the integral and the fact that the integral over S_t is zero, we can conclude that the right-hand sides of (11.17) and (11.18) are indeed equal in this case. □

Example. The mapping of the rectangle $I = \{(r, \varphi) \in \mathbb{R}^2 | \, 0 \le r \le R \wedge 0 \le \varphi \le 2\pi\}$ onto the disk $K = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 \le R^2\}$ given by the formulas

$$x = r\cos\varphi \,, \quad y = r\sin\varphi \,, \qquad (11.21)$$

is not a diffeomorphism: the entire side of the rectangle I on which $r = 0$ maps to the single point $(0, 0)$ under this mapping; the images of the points

$(r, 0)$ and $(r, 2\pi)$ are the same. However, if we consider, for example, the sets $I \setminus \partial I$ and $K \setminus E$, where E is the union of the boundary ∂K of the disk K and the radius ending at $(0, R)$, then the restriction of the mapping (11.21) to the domain $I \setminus \partial I$ turns out to be a diffeomorphism of it onto the domain $K \setminus E$. Hence by Theorem 2, for any function $f \in \mathcal{R}(K)$ we can write

$$\iint\limits_K f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_I f\big(r\cos\varphi, r\sin\varphi\big) r\,\mathrm{d}r\,\mathrm{d}\varphi$$

and, applying Fubini's theorem

$$\iint\limits_K f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^R f\big(r\cos\varphi, r\sin\varphi\big) r\,\mathrm{d}r\ .$$

Relations (11.21) are the well-known formulas for transition from polar coordinates to Cartesian coordinates in the plane.

What has been said can naturally be developed and extended to the polar (spherical) coordinates in $\mathbb{R}^n$ that we studied in Part 1, where we also exhibited the Jacobian of the transition from polar coordinates to Cartesian coordinates in a space $\mathbb{R}^n$ of any dimension.

11.5.8 Problems and Exercises

1. a) Show that Lemma 1 is valid for any smooth mapping $\varphi : D_t \to D_x$ (also see Problem 8 below in this connection).

b) Prove that if D is an open set in $\mathbb{R}^m$ and $\varphi \in C^{(1)}(D, \mathbb{R}^n)$, then $\varphi(D)$ is a set of measure zero in $\mathbb{R}^n$ when $m < n$.

2. a) Verify that the measure of a measurable set E and the measure of its image $\varphi(E)$ under a diffeomorphism φ are connected by the relation $\mu\Big(\varphi(E)\Big) = \theta\mu(E)$, where $\theta = \left[\inf\limits_{t\in E} |\det\varphi'(t)|, \sup\limits_{t\in E} |\det\varphi'(t)|\right]$.

b) In particular, if E is a connected set, there is a point $\tau \in E$ such that $\mu\Big(\varphi(E)\Big) = |\det\varphi'(\tau)|\mu(E)$.

3. a) Show that if formula (11.10) holds for the function $f \equiv 1$, then it holds in general.

b) Carry out the proof of Theorem 1 again, but for the special case $f \equiv 1$, simplifying it for this special situation.

4. Without using Remark 2, carry out the proof of Lemma 3, assuming Lemma 2 is known and that two integrable functions that differ only on a set of measure zero have the same integral.

5. Instead of the additivity of the integral and the accompanying analysis of the measurability of sets, one can use another device for localization when reducing formula (11.10) to its local version (that is to the verification of the formula for a small neighborhood of the points of the domain being mapped). This device is based on the linearity of the integral.

a) If the smooth functions $e_1, \dots, e_k$ are such that $0 \le e_i \le 1$, $i = 1, \dots, k$, and $\sum\limits_i^k e_i(x) \equiv 1$ on D_x, then $\int\limits_{D_x} \left(\sum\limits_{i=1}^k e_i f \right)(x)\,dx = \int\limits_{D_x} f(x)\,dx$ for every function $f \in \mathcal{R}(D_x)$.

b) If supp e_i is contained in the set $U \subset D_x$, then $\int\limits_{D_x} (e_i f)(x)\,dx = \int\limits_U (e_i f)(x)\,dx$.

c) Taking account of Lemmas 3 and 4 and the linearity of the integral, one can derive formula (11.10) from a) and b), if for every open covering $\{U_\alpha\}$ of the compact set $K = \operatorname{supp} f \subset D_x$ we construct a set of smooth functions $e_1, \dots, e_k$ in D_x such that $0 \le e_i \le 1$, $i = 1, \dots, k$, $\sum\limits_{i=1}^k e_i \equiv 1$ on K, and for every function $e_i \in \{e_i\}$ there is a set $U_{\alpha_i} \in \{U_\alpha\}$ such that $\operatorname{supp} e_i \subset U_{\alpha_i}$.

In that case the set of functions $\{e_i\}$ is said to be a *partition of unity on the compact set* K *subordinate to the covering* $\{U_\alpha\}$.

6. This problem contains a scheme for constructing the partition of unity discussed in Problem 5.

a) Construct a function $f \in C^{(\infty)}(\mathbb{R}, \mathbb{R})$ such that $f\Big|_{[-1,1]} \equiv 1$ and $\operatorname{supp} f \subset [-1-\delta, 1+\delta]$, where $\delta > 0$.

b) Construct a function $f \in C^{(\infty)}(\mathbb{R}^n, \mathbb{R})$ with the properties indicated in a) for the unit cube in $\mathbb{R}^n$ and its δ-dilation.

c) Show that for every open covering of the compact set $K \subset \mathbb{R}^n$ there exists a smooth partition of unity on K subordinate to this covering.

d) Extending c), construct a $C^{(\infty)}$-partition of unity in $\mathbb{R}^n$ subordinate to a locally finite open covering of the entire space. (A covering is *locally finite* if every point of the set that is covered, in this case $\mathbb{R}^n$, has a neighborhood that intersects only a finite number of the sets in the covering. For a partition of unity containing an infinite number of functions $\{e_i\}$ we impose the requirement that every point of the set on which this partition is constructed belongs to the support of at most finitely many of the functions $\{e_i\}$. Under this hypothesis no questions arise as to the meaning of the equality $\sum\limits_i e_i \equiv 1$; more precisely, there are no questions as to the meaning of the sum on the left-hand side.)

7. One can obtain a proof of Theorem 1 that is slightly different from the one given above and relies on the possibility of decomposing only a linear mapping into a composition of elementary mappings. Such a proof is closer to the heuristic considerations in Subsect. 11.5.1 and is obtained by proving the following assertions.

a) Verify that under elementary linear mappings $L: \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$(x^1, \dots, x^k, \dots, x^n) \mapsto (x^1, \dots, x^{k-1}, \lambda x^k, x^{k+1}, \dots, x^n)\,,$$

$\lambda \ne 0$, and

$$(x^1, \dots, x^k, \dots, x^n) \mapsto (x^1, \dots, x^{k-1}, x^k + x^j, \dots, x^n)$$

the relation $\mu\Big(L(E)\Big) = |\det L'|\mu(E)$ holds for every measurable set $E \subset \mathbb{R}^n$; then show that this relation holds for every linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$. (Use Fubini's theorem and the possibility of decomposing a linear mapping into a composition of the elementary mappings just exhibited.)

b) Show that if $\varphi : D_t \to D_x$ is a diffeomorphism, then $\mu\Big(\varphi(K)\Big) \le \int\limits_K |\det \varphi'(t)|\,dt$ for every measurable compact set $K \subset D_t$ and its image $\varphi(K)$. (If $a \in D_t$, then $\exists(\varphi'(a))^{-1}$ and in the representation $\varphi(t) = \Big(\varphi'(a) \circ (\varphi'(a))^{-1} \circ \varphi\Big)(t)$ the mapping $\varphi'(a)$ is linear while the transformation $(\varphi'(a))^{-1} \circ \varphi$ is nearly an isometry on a neighborhood of a.)

c) Show that if the function f in Theorem 1 is nonnegative, then $\int\limits_{D_x} f(x)\,dx \le \int\limits_{D_t} \Big((f \circ \varphi)|\det \varphi'|\Big)(t)\,dt$.

d) Applying the preceding inequality to the function $(f \circ \varphi)|\det \varphi'|$ and the mapping $\varphi^{-1} : D_x \to D_t$, show that formula (11.10) holds for a nonnegative function.

e) By representing the function f in Theorem 1 as the difference of integrable nonnegative functions, prove that formula (11.10) holds.

8. Sard's lemma. *Let D be an open set in $\mathbb{R}^n$, let $\varphi \in C^{(1)}(D, \mathbb{R}^n)$, and let S be the set of critical points of the mapping φ. Then $\varphi(S)$ is a set of (Lebesgue) measure zero.*

We recall that a *critical point of a smooth mapping* φ of a domain $D \subset \mathbb{R}^m$ into $\mathbb{R}^n$ is a point $x \in D$ at which $\operatorname{rank} \varphi'(x) < \min\{m, n\}$. In the case $m = n$, this is equivalent to the condition $\det \varphi'(x) = 0$.

a) Verify Sard's lemma for a linear transformation.

b) Let I be an interval in the domain D and $\varphi \in C^{(1)}(D, \mathbb{R}^n)$. Show that there exists a function $\alpha(h)$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that $\alpha(h) \to 0$ as $h \to 0$ and $|\varphi(x + h) - \varphi(x) - \varphi'(x)h| \le \alpha(h)|h|$ for every x, $x + h \in I$.

c) Using b), estimate the deviation of the image $\varphi(I)$ of the interval I under the mapping φ from the same image under the linear mapping $L(x) = \varphi(a) + \varphi'(a)(x - a)$, where $a \in I$.

d) Based on a), b), and c), show that if S is the set of critical points of the mapping φ in the interval I, then $\varphi(S)$ is a set of measure zero.

e) Now finish the proof of Sard's lemma.

f) Using Sard's lemma, show that in Theorem 1 it suffices to require that the mapping φ be a one-to-one mapping of class $C^{(1)}(D_t, D_x)$.

We remark that the version of Sard's lemma given here is a simple special case of a theorem of Sard and Morse, according to which the assertion of the lemma holds even if $D \subset \mathbb{R}^m$ and $\varphi \in C^{(k)}(D, \mathbb{R}^n)$, where $k = \max\{m - n + 1, 1\}$. The quantity k here, as an example of Whitney shows, cannot be decreased for any pair of numbers m and n.

In geometry Sard's lemma is known as the assertion that if $\varphi : D \to \mathbb{R}^n$ is a smooth mapping of an open set $D \subset \mathbb{R}^m$ into $\mathbb{R}^n$, then for almost all points

$x \in \varphi(D)$, the complete pre-image $\varphi^{-1}(x) = M_x$ in D is a surface (manifold) of codimension n in $\mathbb{R}^m$ (that is, $m - \dim M_x = n$ for almost all $x \in D$).

9. Suppose we consider an arbitrary mapping $\varphi \in C^{(1)}(D_t, D_x)$ such that $\det \varphi'(t) \neq 0$ in D_t instead of the diffeomorphism φ of Theorem 1. Let $n(x) = \operatorname{card}\{t \in \operatorname{supp}(f \circ \varphi) \,|\, \varphi(t) = x\}$, that is, $n(x)$ is the number of points of the support of the function $f \circ \varphi$ that map to the point $x \in D_x$ under $\varphi : D_t \to D_x$. The following formula holds:

$$\int\limits_{D_x} (f \cdot n)(x)\,\mathrm{d}x = \int\limits_{D_t} \Big((f \circ \varphi)|\det \varphi'|\Big)(t)\,\mathrm{d}t\,.$$

a) What is the geometric meaning of this formula for $f \equiv 1$?

b) Prove this formula for the special mapping of the annulus $D_t = \{t \in \mathbb{R}^2_t \,|\, 1 < |t| < 2\}$ onto the annulus $D_x = \{x \in \mathbb{R}^2_x \,|\, 1 < |x| < 2\}$ given in polar coordinates (r, φ) and (ρ, θ) in the planes $\mathbb{R}^2_x$ and $\mathbb{R}^2_t$ respectively by the formulas $r = \rho$, $\varphi = 2\theta$.

c) Now try to prove the formula in general.

11.6 Improper Multiple Integrals

11.6.1 Basic Definitions

Definition 1. An *exhaustion* of a set $E \subset \mathbb{R}^m$ is a sequence of measurable sets $\{E_n\}$ such that $E_n \subset E_{n+1} \subset E$ for any $n \in \mathbb{N}$ and $\bigcup\limits_{n=1}^{\infty} E_n = E$.

Lemma. *If $\{E_n\}$ is an exhaustion of a measurable set E, then:*

a) $\lim\limits_{n \to \infty} \mu(E_n) = \mu(E)$;

b) for every function $f \in \mathcal{R}(E)$ the function $f\big|_{E_n}$ also belongs to $\mathcal{R}(E_n)$, and

$$\lim_{n \to \infty} \int\limits_{E_n} f(x)\,\mathrm{d}x = \int\limits_{E} f(x)\,\mathrm{d}x\,.$$

Proof. Since $E_n \subset E_{n+1} \subset E$, it follows that $\mu(E_n) \leq \mu(E_{n+1}) \leq \mu(E)$ and $\lim\limits_{n \to \infty} \mu(E_n) \leq \mu(E)$. To prove *a*) we shall show that the inequality $\lim\limits_{n \to \infty} \mu(E_n) \geq \mu(E)$ also holds.

The boundary ∂E of E has content zero, and hence can be covered by a finite number of open intervals of total content less than any preassigned number $\varepsilon > 0$. Let Δ be the union of all these open intervals. Then the set $E \cup \Delta =: \widetilde{E}$ is open in $\mathbb{R}^m$ and by construction $\widetilde{E}$ contains the closure of E and $\mu(\widetilde{E}) \leq \mu(E) + \mu(\Delta) < \mu(E) + \varepsilon$.

For every set E_n of the exhaustion $\{E_n\}$ the construction just described can be repeated with the value $\varepsilon_n = \varepsilon/2^n$. We then obtain a sequence of open sets $\widetilde{E}_n = E_n \cup \Delta_n$ such that $E_n \subset \widetilde{E}_n$, $\mu(\widetilde{E}_n) \le \mu(E_n)+\mu(\Delta_n) < \mu(E_n)+\varepsilon_n$, and $\bigcup\limits_{n=1}^{\infty} \widetilde{E}_n \supset \bigcup\limits_{n=1}^{\infty} E_n \supset E$.

The system of open sets Δ, $\widetilde{E}_1$, $\widetilde{E}_2,\dots$, forms an open covering of the compact set $\overline{E}$.

Let $\Delta, \widetilde{E}_1, \widetilde{E}_2,\dots,\widetilde{E}_k$ be a finite covering of $\overline{E}$ extracted from this covering. Since $E_1 \subset E_2 \subset \cdots \subset E_k$, the sets Δ, $\Delta_1,\dots$, Δ_k, E_k also form a covering of $\overline{E}$ and hence

$$\mu(E) \le \mu(\overline{E}) \le \mu(E_k) + \mu(\Delta) + \mu(\Delta_1) + \cdots + \mu(\Delta_k) < \mu(E_k) + 2\varepsilon;.$$

It follows from this that $\mu(E) \le \lim\limits_{n\to\infty} \mu(E_n)$.

b) The relation $f\big|_{E_n} \in \mathcal{R}(E_n)$ is well known to us and follows from Lebesgue's criterion for the existence of the integral over a measurable set. By hypothesis $f \in \mathcal{R}(E)$, and so there exists a constant M such that $|f(x)| \le M$ on E. From the additivity of the integral and the general estimate for the integral we obtain

$$\left|\int\limits_E f(x)\,\mathrm{d}x - \int\limits_{E_n} f(x)\,\mathrm{d}x\right| = \left|\int\limits_{E\setminus E_n} f(x)\,\mathrm{d}x\right| \le M\mu(E\setminus E_n)\,.$$

From this, together with what was proved in *a*), we conclude that *b*) does indeed hold. □

Definition 2. Let $\{E_n\}$ be an exhaustion of the set E and suppose the function $f : E \to \mathbb{R}$ is integrable on the sets $E_n \in \{E_n\}$. If the limit

$$\int\limits_E f(x)\,\mathrm{d}x := \lim_{n\to\infty} \int\limits_{E_n} f(x)\,\mathrm{d}x$$

exists and has a value independent of the choice of the sets in the exhaustion of E, this limit is called the *improper integral of f over E*.

The integral sign on the left in this last equality is usually written for any function defined on E, but we say that the integral *exists* or *converges* if the limit in Definition 2 exists. If there is no common limit for all exhaustions of E, we say that the integral of f over E does not exist, or that the integral *diverges*.

The purpose of Definition 2 is to extend the concept of integral to the case of an unbounded integrand or an unbounded domain of integration.

The symbol introduced to denote an improper integral is the same as the symbol for an ordinary integral, and that fact makes the following remark necessary.

Remark 1. If E is a measurable set and $f \in \mathcal{R}(E)$, then the integral of f over E in the sense of Definition 2 exists and has the same value as the proper integral of f over E.

Proof. This is precisely the content of assertion *b*) in the lemma above. □

The set of all exhaustions of any reasonably rich set is immense, and we do not use all exhaustions. The verification that an improper integral converges is often simplified by the following proposition.

Proposition 1. *If a function $f : E \to \mathbb{R}$ is nonnegative and the limit in Definition 2 exists for even one exhaustion $\{E_n\}$ of the set E, then the improper integral of f over E converges.*

Proof. Let $\{E'_k\}$ be a second exhaustion of E into elements on which f is integrable. The sets $E_n^k := E'_k \cap E_n$, $n = 1, 2, \dots$ form an exhaustion of the set E'_k, and so it follows from part *b*) of the lemma that

$$\int\limits_{E'_k} f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int\limits_{E_n^k} f(x)\,\mathrm{d}x \le \lim_{n\to\infty} \int\limits_{E_n} f(x)\,\mathrm{d}x = A\,.$$

Since $f \ge 0$ and $E'_k \subset E'_{k+1} \subset E$, it follows that

$$\exists \lim_{k\to\infty} \int\limits_{E'_k} f(x)\,\mathrm{d}x = B \le A\,.$$

But there is symmetry between the exhaustions $\{E_n\}$ and $\{E'_k\}$, so that $A \le B$ also, and hence $A = B$. □

Example 1. Let us find the improper integral $\iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y$.

We shall exhaust the plane $\mathbb{R}^2$ by the sequence of disks $E_n = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < n^2\}$. After passing to polar coordinates we find easily that

$$\iint\limits_{E_n} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^{n} e^{-r^2} r\,\mathrm{d}r = \pi(1 - e^{-n^2}) \to \pi$$

as $n \to \infty$.

By Proposition 1 we can now conclude that this integral converges and equals π.

One can derive a useful corollary from this result if we now consider the exhaustion of the plane by the squares $E'_n = \{(x,y) \in \mathbb{R}^2 \,|\, |x| \le n \wedge |y| \le n\}$. By Fubini's theorem

$$\iint\limits_{E'_n} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{-n}^{n} \mathrm{d}y \int\limits_{-n}^{n} e^{-(x^2+y^2)}\,\mathrm{d}x = \Bigg(\int\limits_{-n}^{n} e^{-t^2}\,\mathrm{d}t\Bigg)^2\,.$$

By Proposition 1 this last quantity must tend to π as $n \to \infty$. Thus, following Euler and Poisson, we find that

$$\int_{-\infty}^{+\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi}\,.$$

Some additional properties of Definition 2 of an improper integral, which are not completely obvious at first glance, will be given below in Remark 3.

11.6.2 The Comparison Test for Convergence of an Improper Integral

Proposition 2. *Let f and g be functions defined on the set E and integrable over exactly the same measurable subsets of it, and suppose $|f(x)| \le g(x)$ on E. If the improper integral $\int_E g(x)\,\mathrm{d}x$ converges, then the integrals $\int_E |f|(x)\,\mathrm{d}x$ and $\int_E f(x)\,\mathrm{d}x$ also converge.*

Proof. Let $\{E_n\}$ be an exhaustion of E on whose elements both g and f are integrable. It follows from the Lebesgue criterion that the function $|f|$ is integrable on the sets E_n, $n \in \mathbb{N}$, and so we can write

$$\int_{E_{n+k}} |f|(x)\,\mathrm{d}x - \int_{E_n} |f|(x)\,\mathrm{d}x = \int_{E_{n+k}\setminus E_n} |f|(x)\,\mathrm{d}x \le$$
$$\le \int_{E_{n+k}\setminus E_n} g(x)\,\mathrm{d}x = \int_{E_{n+k}} g(x)\,\mathrm{d}x - \int_{E_n} g(x)\,\mathrm{d}x\,,$$

where k and n are any natural numbers. When we take account of Proposition 1 and the Cauchy criterion for the existence of a limit of a sequence, we conclude that the integral $\int_E |f|(x)\,\mathrm{d}x$ converges.

Now consider the functions $f_+ := \frac{1}{2}\big(|f| + f\big)$ and $f_- := \frac{1}{2}\big(|f| - f\big)$. Obviously $0 \le f_+ \le |f|$ and $0 \le f_- \le |f|$. By what has just been proved, the improper integrals of f_+ and f_- over E both converge. But $f = f_+ - f_-$, and hence the improper integral of f over the same set converges as well (and is equal to the difference of the integrals of f_+ and f_-). □

In order to make effective use of Proposition 2 in studying the convergence of improper integrals, it is useful to have a store of standard functions for comparison. In this connection we consider the following example.

Example 2. In the deleted n-dimensional ball of radius 1, $B \subset \mathbb{R}^n$ with its center at 0 removed, consider the function $1/r^\alpha$, where $r = d(0,x)$ is the distance from the point $x \in B \setminus 0$ to the point 0. Let us determine the values of $\alpha \in \mathbb{R}$ for which the integral of $r^{-\alpha}$ over the domain $B \setminus 0$ converges. To do this we construct an exhaustion of the domain by the annular regions $B(\varepsilon) = \{x \in B | \varepsilon < d(0,x) < 1\}$.

Passing to polar coordinates with center at 0, by Fubini's theorem, we obtain

$$\int\limits_{B(\varepsilon)} \frac{dx}{r^{\alpha}(x)} = \int\limits_{S} f(\varphi)\, d\varphi \int\limits_{\varepsilon}^{1} \frac{r^{n-1}\, dr}{r^{\alpha}} = c \int\limits_{\varepsilon}^{1} \frac{dr}{r^{\alpha-n+1}} ,$$

where $d\varphi = d\varphi_1 \dots d\varphi_{n-1}$ and $f(\varphi)$ is a certain product of sines of the angles $\varphi_1, \dots, \varphi_{n-2}$ that appears in the Jacobian of the transition to polar coordinates in $\mathbb{R}^n$, while c is the magnitude of the integral over s, which depends only on n, not on r and ε.

As $\varepsilon \to +0$ the value just obtained for the integral over $B(\varepsilon)$ will have a finite limit if $\alpha < n$. In all other cases this last integral tends to infinity as $\varepsilon \to +0$.

Thus we have shown that the function $\frac{1}{d^{\alpha}(0,x)}$, where d is the distance to the point 0, can be integrated in a deleted neighborhood of 0 only when $\alpha < n$, where n is the dimension of the space.

Similarly one can show that outside the ball B, that is, in a neighborhood of infinity, this same function is integrable in the improper sense only for $\alpha > n$.

Example 3. Let $I = \{x \in \mathbb{R}^n | 0 \le x^i \le 1, i = 1, \dots, n\}$ be the n-dimensional cube and I_k the k-dimensional face of it defined by the conditions $x^{k+1} = \dots = x^n = 0$. On the set $I \setminus I_k$ we consider the function $\frac{1}{d^{\alpha}(x)}$, where $d(x)$ is the distance from $x \in I \setminus I_k$ to the face I_k. Let us determine the values of $\alpha \in \mathbb{R}$ for which the integral of this function over $I \setminus I_k$ converges.

We remark that if $x = (x^1, \dots, x^k, x^{k+1}, \dots, x^n)$ then

$$d(x) = \sqrt{(x^{k+1})^2 + \dots + (x^n)^2} .$$

Let $I(\varepsilon)$ be the cube I from which the ε-neighborhood of the face I_k has been removed. By Fubini's theorem

$$\int\limits_{I(\varepsilon)} \frac{dx}{d^{\alpha}(x)} = \int\limits_{I_k} dx^1 \cdots dx^k \int\limits_{I_{n-k}(\varepsilon)} \frac{dx^{k+1} \cdots dx^n}{\left((x^{k+1})^2 + \dots + (x^n)^2\right)^{\alpha/2}} = \int\limits_{I_{n-k}(\varepsilon)} \frac{du}{|u|^{\alpha}} ,$$

where $u = (x^{k+1}, \dots, x^n)$ and $I_{n-k}(\varepsilon)$ is the face $I_{n-k} \subset \mathbb{R}^{n-k}$ from which the ε-neighborhood of 0 has been removed.

But it is clear on the basis of the experience acquired in Example 1 that the last integral converges only for $\alpha < n - k$. Hence the improper integral under consideration converges only for $\alpha < n - k$, where k is the dimension of the face near which the function may increase without bound.

Remark 2. In the proof of Proposition 2 we verified that the convergence of the integral $|f|$ implies the convergence of the integral of f. It turns out that the converse is also true for an improper integral in the sense of Definition 2, which was not the case previously when we studied improper integrals on the line. In the latter case, we distinguished absolute and nonabsolute (conditional) convergence of an improper integral. To understand right away the essence of the new phenomenon that has arisen in connection with Definition 2, consider the following example.

Example 4. Let the function $f : \mathbb{R}_+ \to \mathbb{R}$ be defined on the set $\mathbb{R}_+$ of nonnegative numbers by the following conditions: $f(x) = \frac{(-1)^{n-1}}{n}$, if $n-1 \le x < n$, $n \in \mathbb{N}$.

Since the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, the integral $\int\limits_0^A f(x)\,\mathrm{d}x$ has a limit as $A \to \infty$ equal to the sum of this series.

However, this series does not converge absolutely, and one can make it divergent to $+\infty$, for example, by rearranging its terms. The partial sums of the new series can be interpreted as the integrals of the function f over the union E_n of the closed intervals on the real line corresponding to the terms of the series. The sets E_n, taken all together, however, form an exhaustion of the domain $\mathbb{R}_+$ on which f is defined.

Thus the improper integral $\int\limits_0^{\infty} f(x)\,\mathrm{d}x$ of the function $f : \mathbb{R}_+ \to \mathbb{R}$ exists in its earlier sense, but not in the sense of Definition 2.

We see that the condition in Definition 2 that the limit be independent of the choice of the exhaustion is equivalent to the independence of the sum of a series on the order of summation. The latter, as we know, is exactly equivalent to absolute convergence.

In practice one nearly always has to consider only special exhaustions of the following type. Let a function $f : D \to \mathbb{R}$ defined in the domain D be unbounded in a neighborhood of some set $E \subset \partial D$. We then remove from D the points lying in the ε-neighborhood of E and obtain a domain $D(\varepsilon) \subset D$. As $\varepsilon \to 0$ these domains generate an exhaustion of D. If the domain is unbounded, we can obtain an exhaustion of it by taking the D-complements of neighborhoods of infinity. These are the special exhaustions we mentioned earlier and studied in the one-dimensional case, and it is these special exhaustions that lead directly to the generalization of the notion of Cauchy principal value of an improper integral to the case of a space of any dimension, which we discussed earlier when studying improper integrals on the line.

11.6.3 Change of Variable in an Improper Integral

In conclusion we obtain the formula for change of variable in improper integrals, thereby making a valuable, although very simple, supplement to Theorems 1 and 2 of Sect. 11.5.

Theorem 1. *Let $\varphi : D_t \to D_x$ be a diffeomorphism of the open set $D_t \subset \mathbb{R}^n_t$ onto the set $D_x \subset \mathbb{R}^n_x$ of the same type, and let $f : D_x \to \mathbb{R}$ be integrable on all measurable compact subsets of D_x. If the improper integral $\int\limits_{D_x} f(x)\,\mathrm{d}x$ converges, then the integral $\int\limits_{D_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t$ also converges and has the same value.*

Proof. The open set $D_t \subset \mathbb{R}^n_t$ can be exhausted by a sequence of compact sets E^k_t, $k \in \mathbb{N}$, contained in $\mathbb{N}$, each of which is the union of a finite number of intervals in $\mathbb{R}^n_t$ (in this connection, see the beginning of the proof of Lemma 1 in Sect. 11.5). Since $\varphi : D_t \to D_x$ is a diffeomorphism, the exhaustion E^k_x of D_x, where $E^k_x = \varphi(E^k_t)$, corresponds to the exhaustion $\{E^k_t\}$ of D_t. Here the sets $E^k_x = \varphi(E^k_t)$ are measurable compact sets in D_x (measurability follows from Lemma 1 of Sect. 11.5). By Proposition 1 of Sect. 11.5 we can write

$$\int\limits_{E^k_x} f(x)\,\mathrm{d}x = \int\limits_{E^k_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t\ .$$

The left-hand side of this equality has a limit by hypothesis as $k \to \infty$. Hence the right-hand side also has the same limit. □

Remark 3. By the reasoning just given we have verified that the integral on the right-hand side of the last equality has the same limit for any exhaustion D_t of the given special type. It is this proven part of the theorem that we shall be using. But formally, to complete the proof of the theorem in accordance with Definition 2 it is necessary to verify that this limit exists for every exhaustion of the domain D_t. We leave this (not entirely elementary) proof to the reader as an excellent exercise. We remark only that one can already deduce the convergence of the improper integral of $|f \circ \varphi|\,|\det \varphi'|$ over the set D_t (see Problem 7).

Theorem 2. *Let $\varphi : D_t \to D_x$ be a mapping of the open sets D_t and D_x. Assume that there are subsets S_t and S_x of measure zero contained in D_t and D_x respectively such that $D_t \setminus S_t$ and $D_x \setminus S_x$ are open sets and φ is a diffeomorphism of the former onto the latter. Under these hypotheses, if the improper integral $\int\limits_{D_x} f(x)\,\mathrm{d}x$ converges, then the integral $\int\limits_{D_t \setminus S_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t$ also converges to the same value. If in addition $|\det \varphi'|$ is defined and bounded on* *compact subsets of* D_t*, then* $(f \circ \varphi)|\det \varphi'|$ *is improperly integrable over the set* D_t*, and the following equality holds:*

$$\int_{D_x} f(x)\,\mathrm{d}x = \int_{D_t \setminus S_t} \big((f \circ \varphi)|\det \varphi'|\big)(t)\,\mathrm{d}t\,.$$

Proof. The assertion is a direct corollary of Theorem 1 and Theorem 2 of Sect. 11.5, provided we take account of the fact that when finding an improper integral over an open set one may restrict consideration to exhaustions that consist of measurable compact sets (see Remark 3).

Example 5. Let us compute the integral $\iint\limits_{x^2+y^2<1} \frac{\mathrm{d}x\,\mathrm{d}y}{(1-x^2-y^2)^\alpha}$, which is an improper integral when $\alpha > 0$, since the integrand is unbounded in that case in a neighborhood of the disk $x^2 + y^2 = 1$.

Passing to polar coordinates, we obtain from Theorem 2

$$\iint\limits_{x^2+y^2<1} \frac{\mathrm{d}x\,\mathrm{d}y}{(1-x^2-y^2)^\alpha} = \iint\limits_{\substack{0<\varphi<2\pi\\0<r<1}} \frac{r\,\mathrm{d}r\,\mathrm{d}\varphi}{(1-r^2)^\alpha}\,.$$

For $\alpha > 0$ this last integral is also improper, but, since the integrand is nonnegative, it can be computed as the limit over the special exhaustion of the rectangle $I = \{(r,\varphi) \in \mathbb{R}^2 | 0 < \varphi < 2\pi \wedge 0 < r < 1\}$ by the rectangles $I_n = \{(r,\varphi) \in \mathbb{R}^2 | 0 < \varphi < 2\pi \wedge 0 < r < 1 - \frac{1}{n}\}$, $n \in \mathbb{N}$. Using Fubini's theorem, we find that

$$\iint\limits_{\substack{0<\varphi<2\pi\\0<r<1}} \frac{r\,\mathrm{d}r\,\mathrm{d}\varphi}{(1-r^2)^\alpha} = \lim_{n\to\infty} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{1-\frac{1}{n}} \frac{r\,\mathrm{d}r}{(1-r^2)^\alpha} = \frac{\pi}{1-\alpha}\,.$$

By the same considerations, one can deduce that the original integral diverges for $\alpha \geq 1$.

Example 6. Let us show that the integral $\iint\limits_{|x|+|y|\geq 1} \frac{\mathrm{d}x\,\mathrm{d}y}{|x|^p+|y|^q}$ converges only under the condition $\frac{1}{p} + \frac{1}{q} < 1$.

Proof. In view of the obvious symmetry it suffices to consider the integral only over the domain D in which $x \geq 0$, $y \geq 0$ and $x + y \geq 1$.

It is clear that the simultaneous conditions $p > 0$ and $q > 0$ are necessary for the integral to converge. Indeed, if $p \leq 0$ for example, we would obtain the following estimate for the integral over the rectangle $I_A = \{(x,y) \in \mathbb{R}^2 | 1 \leq x \leq A \wedge 0 \leq y \leq 1\}$ alone, which is contained in D:

$$\iint\limits_{I_A} \frac{\mathrm{d}x\,\mathrm{d}y}{|x|^p+|y|^q} = \int_1^A \mathrm{d}x \int_0^1 \frac{\mathrm{d}y}{|x|^p+|y|^q} \geq \int_1^A \mathrm{d}x \int_0^1 \frac{\mathrm{d}y}{1+|y|^q} = (A-1)\int_0^1 \frac{\mathrm{d}y}{1+|y|^q}\,,$$

which shows that as $A \to \infty$, this integral increases without bound. Thus from now on we may assume that $p > 0$ and $q > 0$.

The integrand has no singularities in the bounded portion of the domain D, so that studying the convergence of this integral is equivalent to studying the convergence of the integral of the same function over, for example, the portion G of the domain D where $x^p + y^q \geq a > 0$. The number a can be assumed sufficiently large that the curve $x^p + y^q = a$ lies in D for $x \geq 0$ and $y \geq 0$.

Passing to generalized curvilinear coordinates φ using the formulas

$$x = (r\cos^2\varphi)^{1/p}\,, \quad y = (r\sin^2\varphi)^{1/q}\,,$$

by Theorem 2 we obtain

$$\iint\limits_G \frac{\mathrm{d}x\,\mathrm{d}y}{|x|^p + |y|^q} = \frac{2}{p\cdot q} \iint\limits_{\substack{0<\varphi<\pi/2\\ a\leq r<\infty}} \left(r^{\frac{1}{p}+\frac{1}{q}-2}\cos^{\frac{2}{p}-1}\varphi\sin^{\frac{2}{q}-1}\varphi\right)\mathrm{d}r\,\mathrm{d}\varphi\,.$$

Using the exhaustion of the domain $\{(r,\varphi)\in\mathbb{R}^2 | 0<\varphi<\pi/2 \wedge a \leq r < \infty\}$ by intervals $I_{\varepsilon A} = \{(r,\varphi)\in\mathbb{R}^2 | 0 < \varepsilon \leq \varphi \leq \pi/2 - \varepsilon \wedge a \leq r \leq A\}$ and applying Fubini's theorem, we obtain

$$\iint\limits_{\substack{0<\varphi<\pi/2\\ a\leq r<\infty}} \left(r^{\frac{1}{p}+\frac{1}{q}-2}\cos^{\frac{2}{p}-1}\varphi\sin^{\frac{2}{q}-1}\varphi\right)\mathrm{d}r\,\mathrm{d}\varphi =$$

$$= \lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{\pi/2-\varepsilon}\cos^{\frac{2}{p}-1}\varphi\sin^{\frac{2}{q}-1}\varphi\,\mathrm{d}\varphi \lim_{A\to\infty}\int\limits_a^A r^{\frac{1}{p}+\frac{1}{q}-2}\,\mathrm{d}r\,.$$

Since $p > 0$ and $q > 0$, the first of these limits is necessarily finite and the second is finite only when $\frac{1}{p} + \frac{1}{q} < 1$. □

11.6.4 Problems and Exercises

1. Give conditions on p and q under which the integral $\iint\limits_{0<|x|+|y|\leq 1} \frac{\mathrm{d}x\,\mathrm{d}y}{|x|^p+|y|^q}$ converges.

2. a) Does the limit $\lim\limits_{A\to\infty}\int\limits_0^A \cos x^2\,\mathrm{d}x$ exist?

b) Does the integral $\int\limits_{\mathbb{R}^1}\cos x^2\,\mathrm{d}x$ converge in the sense of Definition 2?

c) By verifying that

$$\lim_{n\to\infty}\iint\limits_{|x|\leq n}\sin(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \pi$$

and

$$\lim_{n\to\infty} \iint\limits_{x^2+y^2\le 2\pi n} \sin(x^2+y^2)\,dx\,dy = 0$$

verify that the integral of $\sin(x^2+y^2)$ over the plane $\mathbb{R}^2$ diverges.

3. a) Compute the integral $\int\limits_0^1\int\limits_0^1\int\limits_0^1 \frac{dx\,dy\,dz}{x^p y^q z^r}$.

b) One must be careful when applying Fubini's theorem to improper integrals (but of course one must also be careful when applying it to proper integrals). Show that the integral $\iint\limits_{x\ge1,\,y\ge1} \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy$ diverges, while both of the iterated integrals $\int\limits_1^\infty dx \int\limits_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dy$ and $\int\limits_1^\infty dy \int\limits_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dx$ converge.

c) Prove that if $f \in C(\mathbb{R}^2,\mathbb{R})$ and $f \ge 0$ in $\mathbb{R}^2$, then the existence of either of the iterated integrals $\int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} f(x,y)\,dy$ and $\int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} f(x,y)\,dx$ implies that the integral $\iint\limits_{\mathbb{R}^2} f(x,y)\,dx\,dy$ converges to the value of the iterated integral in question.

4. Show that if $f \in C(\mathbb{R},\mathbb{R})$, then

$$\lim_{h\to0} \frac{1}{\pi}\int\limits_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,dx = f(0)\,.$$

5. Let D be a bounded domain in $\mathbb{R}^n$ with a smooth boundary and S a smooth k-dimensional surface contained in the boundary of D. Show that if the function $f \in C(D,\mathbb{R})$ admits the estimate $|f| < \frac{1}{d^{n-k-\varepsilon}}$, where $d = d(S,x)$ is the distance from $x \in D$ to S and $\varepsilon > 0$, then the integral of f over D converges.

6. As a supplement to Remark 1 show that it remains valid even if the set E is not assumed to be measurable.

7. Let D be an open set in $\mathbb{R}^n$ and let the function $f : D \to \mathbb{R}$ be integrable over any measurable compact set contained in D.

a) Show that if the improper integral of the function $|f|$ over D diverges, then there exists an exhaustion $\{E_n\}$ of D such that each set E_n is an *elementary compact set*, consisting of a finite number of n-dimensional intervals and $\iint\limits_{E_n} |f|(x)\,dx \to +\infty$ as $n \to \infty$.

b) Verify that if the integral of f over a set converges while the integral of $|f|$ diverges, then the integrals of $f_+ = \frac{1}{2}(|f|+f)$ and $f_- = \frac{1}{2}(|f|-f)$ over the set both diverge.

c) Show that the exhaustion $\{E_n\}$ obtained in a) can be distributed in such a way that $\int\limits_{E_{n+1}\setminus E_n} f_+(x)\,dx > \int\limits_{E_n} |f|(x)\,dx$ for all $n \in \mathbb{N}$.

d) Using lower Darboux sums, show that if $\int\limits_E f_+(x)\,dx > A$, then there exists an elementary compact set $F \subset E$ consisting of a finite number of intervals such that $\int\limits_F f(x)\,dx > A$.

e) Deduce from c) and d) that there exists an elementary compact set $F_n \subset E_{n+1} \setminus E_n$ for which $\int\limits_{F_n} f(x)\,dx > \int\limits_{E_n} |f|(x)\,dx + n$.

f) Show using e) that the sets $G_n = F_n \cap E_n$ are elementary compact sets (that is, they consist of a finite number of intervals) contained in D that, taken together, constitute an exhaustion of D, and for which the relation $\int\limits_{G_n} f(x)\,dx \to +\infty$ as $n \to \infty$ holds.

Thus, if the integral of $|f|$ diverges, then the integral of f (in the sense of Definition 2) also diverges.

8. Carry out the proof of Theorem 2 in detail.

9. We recall that if $x = (x^1, \dots, x^n)$ and $\xi = (\xi^1, \dots, \xi^n)$, then $\langle x, \xi \rangle = x^1\xi^1 + \dots + x^n\xi^n$ is the standard inner product in $\mathbb{R}^n$. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix of complex numbers. We denote by $\operatorname{Re} A$ the matrix with elements $\operatorname{Re} a_{ij}$. Writing $\operatorname{Re} A \geq 0$ (resp. $\operatorname{Re} A > 0$) means that $\langle (\operatorname{Re} A)x, x \rangle \geq 0$ (resp. $\langle (\operatorname{Re} A)x, x \rangle > 0$) for every $x \in \mathbb{R}^n$, $x \neq 0$.

a) Show that if $\operatorname{Re} A \geq 0$, then for $\lambda > 0$ and $\xi \in \mathbb{R}^n$ we have

$$\int\limits_{\mathbb{R}^n} \exp\left(-\frac{\lambda}{2}\langle Ax, x \rangle - i\langle x, \xi \rangle\right) dx =$$
$$= \left(\frac{2\pi}{\lambda}\right)^{n/2} (\det A)^{-1/2} \exp\left(-\frac{1}{2\lambda}\langle A^{-1}\xi, \xi \rangle\right).$$

Here the branch of $\sqrt{\det A}$ is chosen as follows:

$$(\det A)^{-1/2} = |\det A|^{-1/2} \exp\left(-i \operatorname{Ind} A\right),$$
$$\operatorname{Ind} A = \frac{1}{2}\sum_{j=1}^{n} \arg \mu_j(A), \qquad |\arg \mu_j(A)| \leq \frac{\pi}{2},$$

where $\mu_j(A)$ are the eigenvalues of A.

b) Let A be a real-valued symmetric nondegenerate $(n \times n)$ matrix. Then for $\xi \in \mathbb{R}^n$ and $\lambda > 0$ we have

$$\int\limits_{\mathbb{R}^n} \exp\left(i\frac{\lambda}{2}\langle Ax, x \rangle - i\langle x, \xi \rangle\right) dx =$$
$$= \left(\frac{2\pi}{\lambda}\right)^{n/2} |\det A|^{-1/2} \exp\left(-\frac{i}{2\lambda}\langle A^{-1}\xi, \xi \rangle\right) \exp\left(\frac{i\pi}{4} \operatorname{sgn} A\right).$$

Here $\operatorname{sgn} A$ is the signature of the matrix, that is,

$$\operatorname{sgn} A = \nu_+(A) - \nu_-(A),$$

where $\nu_+(A)$ is the number of positive eigenvalutes of A and $\nu_-(A)$ the number of negative eigenvalues.

12 Surfaces and Differential Forms in $\mathbb{R}^n$

In this chapter we discuss the concepts of surface, boundary of a surface, and consistent orientation of a surface and its boundary; we derive a formula for computing the area of a surface lying in $\mathbb{R}^n$; and we give some elementary information on differential forms. Mastery of these concepts is very important in working with line and surface integrals, to which the next chapter is devoted.

12.1 Surfaces in $\mathbb{R}^n$

The standard model for a k-dimensional surface is $\mathbb{R}^k$.

Definition 1. A *surface of dimension k* (or *k-dimensional surface* or *k-dimensional manifold*) in $\mathbb{R}^n$ is a subset $S \subset \mathbb{R}^n$ each point of which has a neighborhood[1] in S homeomorphic[2] to $\mathbb{R}^k$.

Definition 2. The mapping $\varphi : \mathbb{R}^k \to U \subset S$ provided by the homeomorphism referred to in the definition of a surface is called a *chart* or a *local chart* of the surface S, $\mathbb{R}^k$ is called the *parameter domain*, and U is the *range* or *domain of action of the chart on the surface S*.

A local chart introduces curvilinear coordinates in U by assigning to the point $x = \varphi(t) \in U$ the set of numbers $t = (t^1, \dots, t^k) \in \mathbb{R}^k$. It is clear from the definition that the set of objects S described by the definition does not change if $\mathbb{R}^k$ is replaced in it by any topological space homeomorphic to $\mathbb{R}^k$. Most often the standard parameter region for local charts is assumed to be an open cube I^k or an open ball B^k in $\mathbb{R}^k$. But this makes no substantial difference.

[1] As before, a neighborhood of a point $x \in S \subset \mathbb{R}^n$ in S is a set $U_S(x) = S \cap U(x)$, where $U(x)$ is a neighborhood of x in $\mathbb{R}^n$. Since we shall be discussing only neighborhoods of a point on a surface in what follows, we shall simplify the notation where no confusion can arise by writing U or $U(x)$ instead of $U_S(x)$.

[2] On $S \subset \mathbb{R}^n$ and hence also on $U \subset S$ there is a unique metric induced from $\mathbb{R}^n$, so that one can speak of a topological mapping of U into $\mathbb{R}^n$.

To carry out certain analogies and in order to make a number of the following constructions easier to visualize, we shall as a rule take a cube I^k as the canonical parameter domain for local charts on a surface. Thus a chart

$$\varphi : I^k \to U \subset S \tag{12.1}$$

gives a local parametric equation $x = \varphi(t)$ for the surface $S \subset \mathbb{R}^n$, and the k-dimensional surface itself thus has the local structure of a deformed standard k-dimensional interval $I^k \subset \mathbb{R}^n$.

The parametric definition of a surface is especially important for computational purposes, as will become clear below. Sometimes one can define the entire surface by a single chart. Such a surface is usually called *elementary*. For example, the graph of a continuous function $f : I^k \to \mathbb{R}$ in $\mathbb{R}^{k+1}$ is an elementary surface. However, elementary surfaces are more the exception than the rule. For example, our ordinary two-dimensional terrestrial sphere cannot be defined by only one chart. An atlas of the surface of the Earth must contain at least two charts (see Problem 3 at the end of this section).

In accordance with this analogy we adopt the following definition.

Definition 3. A set $A(S) := \{\varphi_i : I_i^k \to U_i,\ i \in \mathbb{N}\}$ of local charts of a surface S whose domains of action together cover the entire surface (that is, $S = \bigcup_i U_i$) is called an *atlas of the surface* S.

The union of two atlases of the same surface is obviously also an atlas of the surface.

If no restrictions are imposed on the mappings (12.1), the local parametrizations of the surface, except that they must be homeomorphisms, the surface may be situated very strangely in $\mathbb{R}^n$. For example, it can happen that a surface homeomorphic to a two-dimensional sphere, that is, a topological sphere, is contained in $\mathbb{R}^3$, but the region it bounds is not homeomorphic to a ball (the so-called *Alexander horned sphere*).[3]

To eliminate such complications, which have nothing to do with the questions considered in analysis, we defined a *smooth k-dimensional surface* in $\mathbb{R}^n$ in Sect. 8.7 to be a set $S \subset \mathbb{R}^n$ such that for each $x_0 \in S$ there exists a neighborhood $U(x_0)$ in $\mathbb{R}^n$ and a diffeomorphism $\psi : U(x_0) \to I^n = \{t \in \mathbb{R}^n \mid |t| < 1,\ i = 1, \dots, n\}$ under which the set $U_S(x_0) := S \cap U(x_0)$ maps into the cube $I^k = I^n \cap \{t \in \mathbb{R}^n \mid t^{k+1} = \dots = t^n = 0\}$.

It is clear that a surface that is smooth in this sense is a surface in the sense of Definition 1, since the mappings $x = \psi^{-1}(t^1, \dots, t^k, 0, \dots, 0) = \varphi(t^1, \dots, t^k)$ obviously define a local parametrization of the surface. The converse, as follows from the example of the horned sphere mentioned above, is generally not true, if the mappings φ are merely homeomorphisms. However,

[3] An example of the surface described here was constructed by the American topologist J.W. Alexander (1888–1977).

if the mappings (12.1) are sufficiently regular, the concept of a surface is actually the same in both the old and new definitions.

In essence this has already been shown by Example 8 in Sect. 8.7, but considering the importance of the question, we give a precise statement of the assertion and recall how the answer is obtained.

Proposition. *If the mapping* (12.1) *belongs to class* $C^{(1)}(I^k, \mathbb{R}^n)$ *and has maximal rank at each point of the cube* I^k, *there exists a number* $\varepsilon > 0$ *and a diffeomorphism* $\varphi_\varepsilon : I^m_\varepsilon \to \mathbb{R}^n$ *of the cube* $I^n_\varepsilon := \{t \in \mathbb{R}^n \,|\, |t^i| \leq \varepsilon_i, i = 1, \ldots, n\}$ *of dimension* n *in* $\mathbb{R}^n$ *such that* $\varphi\big|_{I^k \cap I^n_\varepsilon} = \varphi_\varepsilon\big|_{I^k \cap I^n_\varepsilon}$.

In other words, it is asserted that under these hypotheses the mappings (12.1) are locally the restrictions of diffeomorphisms of the full-dimensional cubes I^n_ε to the k-dimensional cubes $I^k_\varepsilon = I^k \cap I^n_\varepsilon$.

Proof. Suppose for definiteness that the first k of the n coordinate functions $x^k = \varphi^i(t^1, \ldots, t^k)$, $i = 1, \ldots, n$, of the mapping $x = \varphi(t)$ are such that $\det\left(\frac{\partial \varphi^i}{\partial t^j}\right)(0) \neq 0$, $i, j = 1, \ldots, k$. Then by the implicit function theorem the relations

$$\begin{cases} x^1 = \varphi^1(t^1, \ldots, t^k)\,, \\ \dots\dots\dots\dots\dots\dots\dots \\ x^k = \varphi^k(t^1, \ldots, t^k)\,, \\ x^{k+1} = \varphi^{k+1}(t^1, \ldots, t^k)\,, \\ \dots\dots\dots\dots\dots\dots\dots \\ x^n = \varphi^n(t^1, \ldots, t^k) \end{cases}$$

near the point $(t_0, x_0) = \big(0, \varphi(0)\big)$ are equivalent to relations

$$\begin{cases} t^1 = f^1(x^1, \ldots, x^k)\,, \\ \dots\dots\dots\dots\dots\dots \\ t^k = f^k(x^1, \ldots, x^k);\,, \\ x^{k+1} = f^{k+1}(x^1, \ldots, x^k)\,, \\ \dots\dots\dots\dots\dots\dots\dots \\ x^n = f^n(x^1, \ldots, x^k)\,. \end{cases}$$

In this case the mapping

$$\begin{cases} t^1 = f^1(x^1,\dots,x^k)\,, \\ \dots\dots\dots\dots\dots\dots \\ t^k = f^k(x^1,\dots,x^k);\,, \\ t^{k+1} = x^{k+1} - f^{k+1}(x^1,\dots,x^k)\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ t^n = x^n - f^n(x^1,\dots,x^k) \end{cases}$$

is a diffeomorphism of a full-dimensional neighborhood of the point $x_0 \in \mathbb{R}^n$. As φ_ε we can now take the restriction to some cube I^n_ε of the diffeomorphism inverse to it. □

By a change of scale, of course, one can arrange to have $\varepsilon = 1$ and a unit cube I^n_ε in the last diffeomorphism.

Thus we have shown that for a smooth surface in $\mathbb{R}^n$ one can adopt the following definition, which is equivalent to the previous one.

Definition 4. The k-dimensional surface in $\mathbb{R}^n$ introduced by Definition 1 is *smooth* (*of class* $C^{(m)}$, $m \geq 1$) if it has an atlas whose local charts are smooth mappings (of class $C^{(m)}$, $m \geq 1$) and have rank k at each point of their domains of definition.

We remark that the condition on the rank of the mappings (12.1) is essential. For example, the analytic mapping $\mathbb{R} \ni t \mapsto (x^1, x^2) \in \mathbb{R}^2$ defined by $x^1 = t^2$, $x^2 = t^3$ defines a curve in the plane $\mathbb{R}^2$ having a cusp at $(0,0)$. It is clear that this curve is not a smooth one-dimensional surface in $\mathbb{R}^2$, since the latter must have a tangent (a one-dimensional tangent plane) at each point.[4]

Thus, in particular one should not conflate the concept of a smooth path of class $C^{(m)}$ and the concept of a smooth curve of class $C^{(m)}$.

In analysis, as a rule, we deal with rather smooth parametrizations (12.1) of rank k. We have verified that in this case Definition 4 adopted here for a smooth surface agrees with the one considered earlier in Sect. 8.7. However, while the previous definition was intuitive and eliminated certain unnecessary complications immediately, the well-known advantage of Definition 4 of a surface, in accordance with Definition 1, is that it can easily be extended to the definition of an abstract manifold, not necessarily embedded in $\mathbb{R}^n$. For the time being, however, we shall be interested only in surfaces in $\mathbb{R}^n$.

Let us consider some examples of such surfaces.

Example 1. We recall that if $F^i \in C^{(m)}(\mathbb{R}^n, \mathbb{R})$, $i = 1, \dots, n-k$, is a set of smooth functions such that the system of equations

[4] For the tangent plane see Sect. 8.7.

$$\begin{cases} F^1(x^1,\ldots,x^k,x^{k+1},\ldots,x^n)=0\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F^{n-k}(x^1,\ldots,x^k,x^{k+1},\ldots,x^n)=0 \end{cases} \tag{12.2}$$

has rank $n-k$ at each point in the set S of its solutions, then either this system has no solutions at all or the set of its solutions forms a k-dimensional $C^{(m)}$-smooth surface S in $\mathbb{R}^n$.

Proof. We shall verify that if $S \neq \varnothing$, then S does indeed satisfy Definition 4. This follows from the implicit function theorem, which says that in some neighborhood of each point $x_0 \in S$ the system (12.2) is equivalent, up to a relabeling of the variables, to a system

$$\begin{cases} x^{k+1}=f^{k+1}(x^1,\ldots,x^k)\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x^n=f^n(x^1,\ldots,x^k) \end{cases},$$

where $f^{k+1},\ldots,f^n \in C^{(m)}$. By writing this last system as

$$\begin{cases} x^1 &= t^1\,, \\ \ldots & \ldots\ \ldots \\ x^k &= t^k\,, \\ x^{k+1} &= f^{k+1}(t^1,\ldots,t^k)\,, \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ x^n &= f^n(t^1,\ldots,t^k)\,, \end{cases}$$

we arrive at a parametric equation for the neighborhood of the point $x_0 \in S$ on S. By an additional transformation one can obviously turn the domain into a canonical domain, for example, into I^k and obtain a standard local chart (12.1). □

Example 2. In particular, the sphere defined in $\mathbb{R}^n$ by the equation

$$(x^1)^2+\cdots+(x^n)^2=r^2 \qquad (r>0) \tag{12.3}$$

is an $(n-1)$-dimensional smooth surface in $\mathbb{R}^n$ since the set S of solutions of Eq. (12.3) is obviously nonempty and the gradient of the left-hand side of (12.3) is nonzero at each point of S.

When $n = 2$, we obtain the circle in $\mathbb{R}^2$ given by

$$(x^1)^2 + (x^2)^2 = r^2 ,$$

which can easily be parametrized locally by the polar angle θ using the polar coordinates

$$\begin{cases} x^1 = r\cos\theta , \\ x^2 = r\sin\theta . \end{cases}$$

For fixed $r > 0$ the mapping $\theta \mapsto (x^1, x^2)(\theta)$ is a diffeomorphism on every interval of the form $\theta_0 < \theta < \theta_0 + 2\pi$, and two charts (for example, those corresponding to values $\theta_0 = 0$ and $\theta_0 = -\pi$) suffice to produce an atlas of the circle. We could not get by with one canonical chart (12.1) here because a circle is compact, in contrast to $\mathbb{R}^1$ or $I^1 = B^1$, and compactness is invariant under topological mappings.

Polar (spherical) coordinates can also be used to parametrize the two-dimensional sphere

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = r^2$$

in $\mathbb{R}^3$. Denoting by ψ the angle between the direction of the vector (x^1, x^2, x^3) and the positive x^3-axis (that is, $0 \le \psi \le \pi$) and by φ the polar angle of the projection of the radius-vector (x^1, x^2, x^3) onto the (x^1, x^2)-plane, we obtain

$$\begin{cases} x^3 &= r\cos\psi , \\ x^2 &= r\sin\psi\sin\varphi , \\ x^1 &= r\sin\psi\cos\varphi . \end{cases}$$

In general polar coordinates $(r, \theta_1, \dots, \theta_{n-1})$ in $\mathbb{R}^n$ are introduced via the relations

$$\begin{cases} x^1 &= r\cos\theta_1 , \\ x^2 &= r\sin\theta_1\cos\theta_2 , \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x^{n-1} &= r\sin\theta_1\sin\theta_2\cdot\ldots\cdot\sin\theta_{n-2}\cos\theta_{n-1} , \\ x^n &= r\sin\theta_1\sin\theta_2\cdot\ldots\cdot\sin\theta_{n-1}\sin\theta_{n-1} . \end{cases} \tag{12.4}$$

We recall the Jacobian

$$J = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdot\ldots\cdot\sin\theta_{n-2} \tag{12.5}$$

for the transition (12.4) from polar coordinates $(r, \theta_1, \dots, \theta_{n-1})$ to Cartesian coordinates $(x^1, \dots, x^n)$ in $\mathbb{R}^n$. It is clear from the expression for the Jacobian that it is nonzero if, for example, $0 < \theta_i < \pi$, $i = 1, \dots, n-2$, and $r > 0$. Hence, even without invoking the simple geometric meaning of the parameters $\theta_1, \dots, \theta_{n-1}$, one can guarantee that for a fixed $r > 0$ the mapping $(\theta_1, \dots, \theta_{n-1}) \mapsto (x^1, \dots, x^n)$, being the restriction of a local diffeomorphism $(r, \theta_1, \dots, \theta_{n-1}) \mapsto (x^1, \dots, x^n)$ is itself a local diffeomorphism. But the sphere is homogeneous under the group of orthogonal transformations of $\mathbb{R}^n$, so that the possibility of constructing a local chart for a neighborhood of any point of the sphere now follows.

Example 3. The cylinder

$$(x^1)^2 + \cdots + (x^k)^2 = r^2 \quad (r > 0)\ ,$$

for $k < n$ is an $(n-1)$-dimensional surface in $\mathbb{R}^n$ that is the direct product of the $(k-1)$-dimensional sphere in the plane of the variables $(x^1, \dots, x^k)$ and the $(n-k)$-dimensional plane of the variables $(x^{k+1}, \dots, x^n)$.

A local parametrization of this surface can obviously be obtained if we take the first $k-1$ of the $n-1$ parameters $(t^1, \dots, t^{n-1})$ to be the polar coordinates $\theta_1, \dots, \theta_{k-1}$ of a point of the $(k-1)$-dimensional sphere in $\mathbb{R}^k$ and set $t^k, \dots, t^{n-1}$ equal to $x^{k+1}, \dots, x^n$ respectively.

Example 4. If we take a curve (a one-dimensional surface) in the plane $x = 0$ of $\mathbb{R}^3$ endowed with Cartesian coordinates (x, y, z), and the curve does not intersect the z-axis, we can rotate the curve about the z-axis and obtain a 2-dimensional surface. The local coordinates can be taken as the local coordinates of the original curve (the meridian) and, for example, the angle of revolution (a local coordinate on a parallel of latitude).

In particular, if the original curve is a circle of radius a with center at $(b, 0, 0)$, for $a < b$ we obtain the two-dimensional torus (Fig. 12.1). Its parametric equation can be represented in the form

$$\begin{cases} x = (b + a\cos\psi)\cos\varphi\ , \\ y = (b + a\cos\psi)\sin\varphi\ , \\ z = a\sin\psi\ , \end{cases}$$

where ψ is the angular parameter on the original circle – the meridian – and φ is the angle parameter on a parallel of latitude.

It is customary to refer to any surface homeomorphic to the torus of revolution just constructed as a *torus* (more precisely, a *two-dimensional torus*). As one can see, a two-dimensional torus is the direct product of two circles. Since a circle can be obtained from a closed interval by gluing together (identifying) its endpoints, a torus can be obtained from the direct product of two closed intervals (that is, a rectangle) by gluing the opposite sides together at corresponding points (Fig. 12.2).

In essence, we have already made use of this device earlier when we established that the configuration space of a double pendulum is a two-dimensional torus, and that a path on the torus corresponds to a motion of the pendulum.

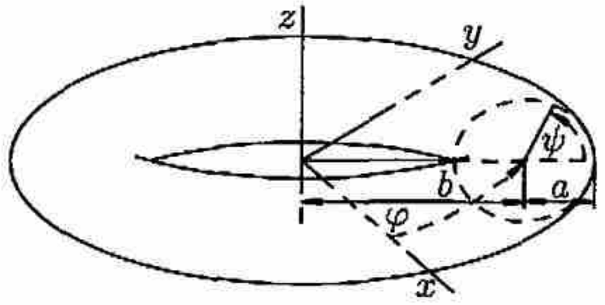

Fig. 12.1.

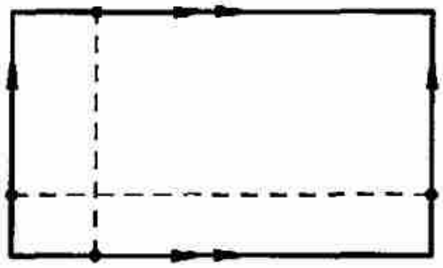

Fig. 12.2.

Example 5. If a flexible ribbon (rectangle) is glued along the arrows shown in Fig. 12.3,a, one can obtain an annulus (Fig. 12.3,c) or a cylindrical surface (Fig. 12.3,b), which are the same from a topological point of view. (These two surfaces are homeomorphic.) But if the ribbon is glued together along the arrows shown in Fig. 12.4a, we obtain a surface in $\mathbb{R}^3$ (Fig. 12.4,b) called a *Möbius band.*[5]

Local coordinates on this surface can be naturally introduced using the coordinates on the plane in which the original rectangle lies.

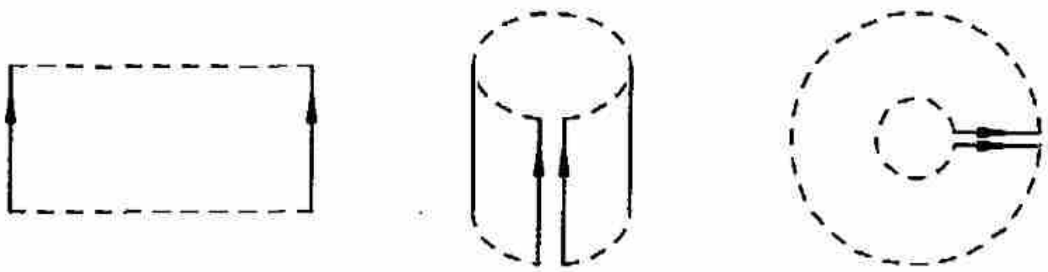

Fig. 12.3.

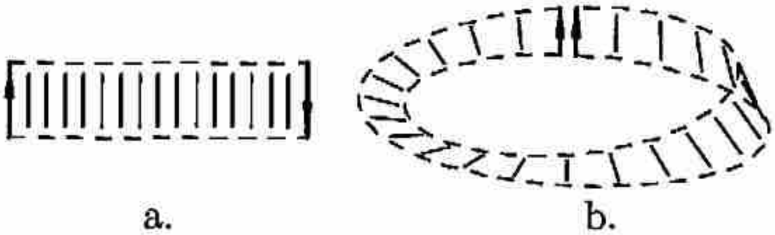

Fig. 12.4.

Example 6. Comparing the results of Examples 4 and 5 in accordance with the natural analogy, one can now prescribe how to glue a rectangle (Fig. 12.5,a) that combines elements of the torus and elements of the Möbius band. But, just as it was necessary to go outside $\mathbb{R}^2$ in order to glue the

[5] A.F. Möbius (1790–1868) – German mathematician and astronomer.

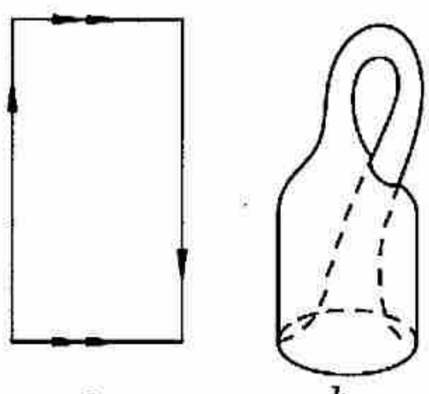

Fig. 12.5.

Möbius band without tearing or self-intersections, the gluing prescribed here cannot be carried out in $\mathbb{R}^3$. However, this can be done in $\mathbb{R}^4$, resulting in a surface in $\mathbb{R}^4$ usually called the *Klein bottle.*[6] An attempt to depict this surface has been undertaken in Fig. 12.5,b.

This last example gives some idea of how a surface can be intrinsically described more easily than the same surface lying in a particular space $\mathbb{R}^n$. Moreover, many important surfaces (of different dimensions) originally arise not as subsets of $\mathbb{R}^n$, but, for example, as the phase spaces of mechanical systems or the geometric image of continuous transformation groups of automorphisms, as the quotient spaces with respect to groups of automorphisms of the original space, and so on, and so forth. We confine ourselves for the time being to these introductory remarks, waiting to make them more precise until Chapter 15, where we shall give a general definition of a surface not necessarily lying in $\mathbb{R}^n$. But already at this point, before the definition has even been given, we note that by a well-known theorem of Whitney[7] any k-dimensional surface can be mapped homeomorphically onto a surface lying in $\mathbb{R}^{2k+1}$. Hence in considering surfaces in $\mathbb{R}^n$ we really lose nothing from the point of view of topological variety and classification. These questions, however, are somewhat off the topic of our modest requirements in geometry.

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[6] F.Ch. Klein (1849–1925) – outstanding German mathematician, the first to make a rigorous investigation of non-Euclidean geometry. An expert in the history of mathematics and one of the organizers of the "Encyclopädie der mathematischen Wissenschaften."

[7] H. Whitney (1907–1989) – American topologist, one of the founders of the theory of fiber bundles.

12.1.1 Problems and Exercises

1. For each of the sets E_α given by the conditions

$$\begin{aligned} E_\alpha &= \{(x,y) \in \mathbb{R}^2 | x^2 - y^2 = \alpha\}\,, \\ E_\alpha &= \{(x,y,z) \in \mathbb{R}^3 | x^2 - y^2 = \alpha\}\,, \\ E_\alpha &= \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = \alpha\}\,, \\ E_\alpha &= \{z \in \mathbb{C} | \, |z^2 - 1| = \alpha\}\,, \end{aligned}$$

depending on the value of the parameter $\alpha \in \mathbb{R}$, determine

a) whether E_α is a surface;

b) if so, what the dimension of E_α is;

c) whether E_α is connected.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping satisfying the condition $f \circ f = f$.

a) Show that the set $f(\mathbb{R}^n)$ is a smooth surface in $\mathbb{R}^n$.

b) By what property of the mapping f is the dimension of this surface determined?

3. Let $e_0, e_1, \dots, e_n$ be an orthonormal basis in the Euclidean space $\mathbb{R}^{n+1}$, let $x = x^0 e_0 + x^1 e_1 + \dots + x^n e_n$, let $\{x\}$ be the point $(x^0, x^1, \dots, x^n)$, and let $e_1, \dots, e_n$ be a basis in $\mathbb{R}^n \subset \mathbb{R}^{n+1}$.

The formulas

$$\psi_1 = \frac{x - x^0 e_0}{1 - x^0} \ \text{ for } x \neq e_0, \quad \psi_2 = \frac{x - x^0 e_0}{1 + x^0} \ \text{ for } x \neq -e_0$$

define the stereographic projections

$$\psi_1 : S^n \setminus \{e_0\} \to \mathbb{R}^n, \quad \psi : S^n \setminus \{-e_0\} \to \mathbb{R}^n$$

from the points $\{e_0\}$ and $\{-e_0\}$ respectively.

a) Determine the geometric meaning of these mappings.

b) Verify that if $t \in \mathbb{R}^n$ and $t \neq 0$, then $(\psi_2 \circ \psi_1^{-1})(t) = \frac{t}{|t|^2}$, where $\psi_1^{-1} = (\psi_1|_{S_n \setminus \{e_0\}})^{-1}$.

c) Show that the two charts $\psi_1^{-1} = \varphi_1 : \mathbb{R}^n \to S^n \setminus \{e_0\}$ and $\psi_2^{-1} = \varphi_2 : \mathbb{R}^n \to S^n \setminus \{-e_0\}$ form an atlas of the sphere $S^n \subset \mathbb{R}^{n+1}$.

d) Prove that every atlas of the sphere must have at least two charts.

12.2 Orientation of a Surface

We recall first of all that the transition from one frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ in $\mathbb{R}^n$ to a second frame $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n$ is effected by means of the square matrix obtained from the expansions $\tilde{\mathbf{e}}_j = a_j^i \mathbf{e}_i$. The determinant of this matrix is always nonzero, and the set of all frames divides into two equivalence classes, each class containing all possible frames such that for any two of them the determinant of the transition matrix is positive. Such equivalence classes are called *orientation classes of frames* in $\mathbb{R}^n$.

To define an orientation means to fix one of these orientation classes. Thus, the *oriented space* $\mathbb{R}^n$ is the space $\mathbb{R}^n$ itself together with a fixed orientation class of frames. To specify the orientation class it suffices to exhibit any of the frames in it, so that one can also say that the oriented space $\mathbb{R}^n$ is $\mathbb{R}^n$ together with a fixed frame in it.

A frame in $\mathbb{R}^n$ generates a coordinate system in $\mathbb{R}^n$, and the transition from one such coordinate system to another is effected by the matrix (a_i^j) that is the transpose of the matrix (a_j^i) that connects the two frames. Since the determinants of these two matrices are the same, everything that was said above about orientation can be repeated on the level of *orientation classes of coordinate systems in* $\mathbb{R}^n$, placing in one class all the coordinate systems such that the transition matrix between any two systems in the same class has a positive Jacobian.

Both of these essentially identical approaches to describing the concept of an orientation in $\mathbb{R}^n$ will also manifest themselves in describing the orientation of a surface, to which we now turn.

We recall, however, another connection between coordinates and frames in the case of curvilinear coordinate systems, a connection that will be useful in what is to follow.

Let G and D be diffeomorphic domains lying in two copies of the space $\mathbb{R}^n$ endowed with Cartesian coordinates $(x^1, \dots, x^n)$ and $(t^1, \dots, t^n)$ respectively. A diffeomorphism $\varphi : D \to G$ can be regarded as the introduction of curvilinear coordinates $(t^1, \dots, t^n)$ into the domain G via the rule $x = \varphi(t)$, that is, the point $x \in G$ is endowed with the Cartesian coordinates $(t^1, \dots, t^n)$ of the point $t = \varphi^{-1}(x) \in D$. If we consider a frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ of the tangent space $T\mathbb{R}^n_t$ at each point $t \in D$ composed of the unit vectors along the coordinate directions, a field of frames arises in D, which can be regarded as the translations of the orthogonal frame of the original space $\mathbb{R}^n$ containing D, parallel to itself, to the points of D. Since $\varphi : D \to G$ is a diffeomorphism, the mapping $\varphi'(t) : TD_t \to TG_{x=\varphi(t)}$ of tangent spaces effected by the rule $TD_t \ni \mathbf{e} \mapsto \varphi'(t)\mathbf{e} = \boldsymbol{\xi} \in TG_x$, is an isomorphism of the tangent spaces at each point t. Hence from the frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ in TD_t we obtain a frame $\boldsymbol{\xi}_1 = \varphi'(t)\mathbf{e}_1, \dots, \boldsymbol{\xi}_n = \varphi'(t)\mathbf{e}_n$ in TG_x, and the field of frames on D transforms into a field of frames on G (see Fig. 12.6). Since $\varphi \in C^{(1)}(D, G)$, the vector field $\boldsymbol{\xi}(x) = \boldsymbol{\xi}\big(\varphi(t)\big) = \varphi'(t)\mathbf{e}(t)$ is continuous in G if the vector field $\mathbf{e}(t)$ is continuous in D. Thus every continuous field of frames (consisting of n continuous vector fields) transforms under a diffeomorphism to a continuous field of frames. Now let us consider a pair of diffeomorphisms $\varphi_i : D_i \to G$, $i = 1, 2$, which introduce two systems of curvilinear coordinates $(t_1^1, \dots, t_1^n)$ and $(t_2^1, \dots, t_2^n)$ into the same domain G. The mutually inverse diffeomorphisms $\varphi_2^{-1} \circ \varphi_1 : D_1 \to D_2$ and $\varphi_1^{-1} \circ \varphi_2 : D_2 \to D_1$ provide

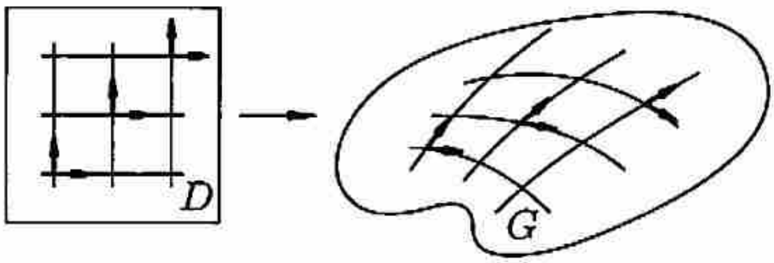


Fig. 12.6.

mutual transitions between these coordinate systems. The Jacobians of these mappings at corresponding points of D_1 and D_2 are mutually inverse to each other and consequently have the same sign. If the domain G (and together with it D_1 and D_2) is connected, then by the continuity and nonvanishing of the Jacobians under consideration, they have the same sign at all points of the domains D_1 and D_2 respectively.

Hence the set of all curvilinear coordinate systems introduced in a connected domain G by this method divide into exactly two equivalence classes when each class is assigned systems whose mutual transitions are effected with a positive Jacobian. Such equivalence classes are called the *orientation classes of curvilinear coordinate systems* in G.

To define an orientation in G means by definition to fix an orientation class of its curvilinear coordinate systems.

It is not difficult to verify that curvilinear coordinate systems belonging to the same orientation class generate continuous fields of frames in G (as described above) that are in the same orientation class of the tangent space TG_x at each point $x \in G$. It can be shown in general that, if G is connected, the continuous fields of frames on G divide into exactly two equivalence classes if each class is assigned the fields whose frames belong to the same orientation class of frames of the space TG_x at each point $x \in G$ (in this connection, see Problems 3 and 4 at the end of this section).

Thus the same orientation of a domain G can be defined in two completely equivalent ways: by exhibiting a curvilinear coordinate system in G, or by defining any continuous field of frames in G, all belonging to the same orientation class as the field of frames generated by this coordinate system.

It is now clear that the orientation of a connected domain G is completely determined if a frame that orients TG_x is prescribed at even one point $x \in G$. This circumstance is widely used in practice. If such an *orienting frame* is defined at some point $x_0 \in G$, and a curvilinear coordinate system $\varphi : D \to G$ is taken in G, then after constructing the frame induced by this coordinate system in TG_{x_0}, we compare it with the orienting frame in TG_x. If the two frames both belong to the same orientation class of TG_{x_0}, we regard the curvilinear coordinates as defining the same orientation on G as the orienting frame. Otherwise, we regard them as defining the opposite orientation.

If G is an open set, not necessarily connected, since what has just been said is applicable to any connected component of G, it is necessary to define an orienting frame in each component of G in order to orient G. Hence, if there are m components, the set G admits 2^m different orientations.

What has just been said about the orientation of a domain $G \subset \mathbb{R}^n$ can be repeated verbatim if instead of the domain G we consider a smooth k-dimensional surface S in $\mathbb{R}^n$ defined by a single chart (see Fig. 12.7). In this case the curvilinear coordinate systems on S also divide naturally into two orientation classes in accordance with the sign of the Jacobian of their mutual transition transformations; fields of frames also arise on S; and the orientation can also be defined by an orienting frame in some tangent plane TS_{x_0} to S.

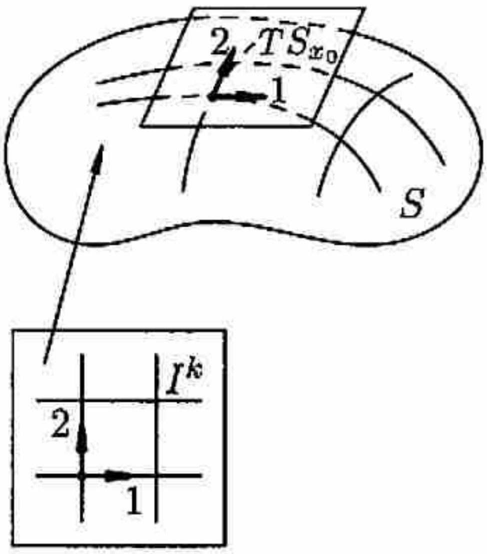


Fig. 12.7.

The only new element that arises here and requires verification is the implicitly occurring proposition that follows.

Proposition 1. *The mutual transitions from one curvilinear coordinate system to another on a smooth surface $S \subset \mathbb{R}^n$ are diffeomorphisms of the same degree of smoothness as the charts of the surface.*

Proof. In fact, by the proposition in Sect. 12.1, we can regard any chart $I^k \to U \subset S$ locally as the restriction to $I^k \cap O(t)$ of a diffeomorphism $\mathcal{F} : O(t) \to O(x)$ from some n-dimensional neighborhood $O(t)$ of the point $t \in I^k \subset \mathbb{R}^n$ to an n-dimensional neighborhood $O(x)$ of $x \in S \subset \mathbb{R}^n$, $\mathcal{F}$ being of the same degree of smoothness as φ. If now $\varphi_1 : I_1^k \to U_1$ and $\varphi_2 : I_2^k \to U_2$ are two such charts, then the action of the mapping $\varphi_2^{-1} \circ \varphi_1$ (the transition from the first coordinate system to the second) which arises in the common domain of action can be represented locally as $\varphi_2^{-1} \circ \varphi_1(t^1, \dots, t^k) = \mathcal{F}_2^{-1} \circ \mathcal{F}(t^1, \dots, t^k, 0, \dots, 0)$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the corresponding diffeomorphisms of the n-dimensional neighborhoods. □

We have studied all the essential components of the concept of an orientation of a surface using the example of an elementary surface defined by a single chart. We now finish up this business with the final definitions relating to the case of an arbitrary smooth surface in $\mathbb{R}^n$.

Let S be a smooth k-dimensional surface in $\mathbb{R}^n$, and let $\varphi_i : I_i^k \to U_i$, $\varphi_j : I_j^k \to U_j$ be two local charts of the surface S whose domains of action intersect, that is, $U_i \cap U_j \neq \varnothing$. Then between the sets $I_{ij}^k = \varphi_i^{-1}(U_j)$ and $I_{ji}^k = \varphi_j^{-1}(U_i)$, as was just proved, there are natural mutually inverse diffeomorphisms $\varphi_{ij} : I_{ij}^k \to I_{ji}^k$ and $\varphi_{ji} : I_{ji}^k \to I_{ij}^k$ that realize the transition from one local curvilinear coordinate system on S to the other.

Definition 1. Two local charts of a surface are *consistent* if their domains of action either do not intersect, or have a nonempty intersection for which the mutual transitions are effected by diffeomorphisms with positive Jacobian in their common domain of action.

Definition 2. An atlas of a surface is an *orienting atlas of the surface* if it consists of pairwise consistent charts.

Definition 3. A surface is *orientable* if it has an orienting atlas. Otherwise it is *nonorientable*.

In contrast to domains of $\mathbb{R}^n$ or elementary surfaces defined by a single chart, an arbitrary surface may turn out to be nonorientable.

Example 1. The Möbius band, as one can verify (see Problems 2 and 3 at the end of this section), is a nonorientable surface.

Example 2. The Klein bottle is also a nonorientable surface, since it contains a Möbius band. This last fact can be seen immediately from the construction of the Klein bottle shown in Fig. 12.5.

Example 3. A circle and in general a k-dimensional sphere are orientable, as can be proved by exhibiting directly an atlas of the sphere consisting of consistent charts (see Example 2 of Sect. 12.1).

Example 4. The two-dimensional torus studied in Example 4 of Sect. 12.1 is also an orientable surface. Indeed, using the parametric equations of the torus exhibited in Example 4 of Sect. 12.1, one can easily exhibit an orienting atlas for it.

We shall not go into detail, since a more visualizable method of controlling the orientability of sufficiently simple surfaces will be exhibited below, making it easy to verify the assertions in Examples 1–4.

The formal description of the concept of orientation of a surface will be finished if we add Definitions 4 and 5 below to Definitions 1, 2, and 3.

Two orienting atlases of a surface are *equivalent* if their union is also an orienting atlas of the surface.

This relation is indeed an equivalence relation between orienting atlases of an orientable surface.

Definition 4. An equivalence class of orienting atlases of a surface under this relation is called an *orientation class of atlases* or simply an *orientation of the surface.*

Definition 5. An *oriented surface* is a surface with a fixed orientation class of atlases (that is, a fixed orientation of the surface).

Thus *orienting a surface* means exhibiting a particular orientation class of orienting atlases of the surface by some means or other.

Some special manifestations of the following proposition are already familiar to us.

Proposition 2. *There exist precisely two orientations on a connected orientable surface.*

They are usually called *opposite orientations.*

The proof of Proposition 2 will be given in Subsect. 15.2.3.

If an orientable surface is connected, an orientation of it can be defined by specifying any local chart of the surface or an orienting frame in any of its tangent planes. This fact is widely used in practice.

When a surface has more than one connected component, such a local chart or frame is naturally to be exhibited in each component.

The following way of defining an orientation of a surface embedded in a space that already carries an orientation is widely used in practice. Let S be an orientable $(n-1)$-dimensional surface embedded in the Euclidean space $\mathbb{R}^n$ with a fixed orienting frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ in $\mathbb{R}^n$. Let TS_x be the $(n-1)$-dimensional plane tangent to S at $x \in S$, and $\mathbf{n}$ the vector orthogonal to TS_x, that is, the vector normal to the surface S at x. If we agree that for the given vector $\mathbf{n}$ the frame $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}$ is to be chosen in TS_x so that the frames $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ and $(\mathbf{n}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}) = (\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n)$ belong to the same orientation class on $\mathbb{R}^n$, then, as one can easily see, such frames $(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n)$ of the plane TS_x will themselves all turn out to belong to the same orientation class for this plane. Hence in this case defining an orientation class for TS_x and along with it an orientation on a connected orientable surface can be done by defining the normal vector $\mathbf{n}$ (Fig. 12.8).

It is not difficult to verify (see Problem 4) that the orientability of an $(n-1)$-dimensional surface embedded in the Euclidean space $\mathbb{R}^n$ is equivalent to the existence of a continuous field of nonzero normal vectors on the surface.

Hence, in particular, the orientability of the sphere and the torus follow obviously, as does the nonorientability of the Möbius band, as was stated in Examples 1–4.

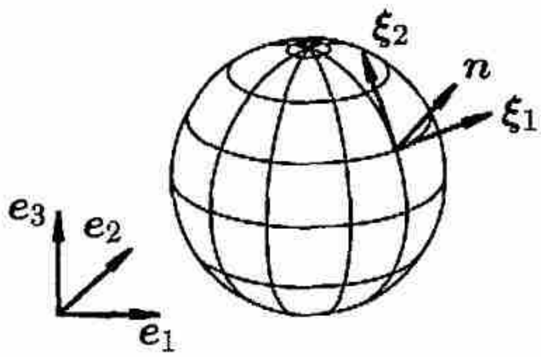


Fig. 12.8.

In geometry the connected $(n-1)$-dimensional surfaces in the Euclidean space $\mathbb{R}^n$ on which there exists a (single-valued) continuous field of unit normal vectors are called *two-sided.*

Thus, for example, a sphere, torus, or plane in $\mathbb{R}^3$ is a two-sided surface, in contrast to the Möbius band, which is a *one-sided surface* in this sense.

To finish our discussion of the concept of orientation of a surface, we make several remarks on the practical use of this concept in analysis.

In the computations that are connected in analysis with oriented surfaces in $\mathbb{R}^n$ one usually finds first some local parametrization of the surface S without bothering about orientation. In some tangent plant TS_x to the surface one then constructs a frame $\xi_1, \dots, \xi_{n-1}$ consisting of (velocity) vectors tangent to the coordinate lines of a chosen curvilinear coordinate system, that is, the orienting frame induced by this coordinate system.

If the space $\mathbb{R}^n$ has been oriented and an orientation of S has been defined by a field of normal vectors, one chooses the vector $\mathbf{n}$ of the given field at the point x and compares the frame $\mathbf{n}, \xi_1, \dots, \xi_{n-1}$ with the frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ that orients the space. If these are in the same orientation class, the local chart defines the required orientation of the surface in accordance with our convention. If these two frames are inconsistent, the chosen chart defines an orientation of the surface opposite to the one prescribed by the normal $\mathbf{n}$.

It is clear that when there is a local chart of an $(n-1)$-dimensional surface, one can obtain a local chart of the required orientation (the one prescribed by the fixed normal vector $\mathbf{n}$ to the two-sided hypersurface embedded in the oriented space $\mathbb{R}^n$) by a simple change in the order of the coordinates.

In the one-dimensional case, in which a surface is a curve, the orientation is more often defined by the tangent vector to the curve at some point; in that case we often say *the direction of motion along the curve* rather than "the orientation of the curve."

If an orienting frame has been chosen in $\mathbb{R}^2$ and a closed curve is given, the *positive direction of circuit*around the domain D bounded by the curve is taken to be the direction such that the frame $\mathbf{n}$, v, where n is the exterior normal to the curve with respect to D and v is the velocity of the motion, is consistent with the orienting frame in $\mathbb{R}^2$.

This means, for example, that for the traditional frame drawn in the plane a positive circuit is "counterclockwise", in which the domain is always "on the left".

In this connection the orientation of the plane itself or of a portion of the plane is often defined by giving the positive direction along some closed curve, usually a circle, rather than a frame in $\mathbb{R}^2$.

Defining such a direction amounts to exhibiting the direction of shortest rotation from the first vector in the frame until it coincides with the second, which is equivalent to defining an orientation class of frames on the plane.

12.2.1 Problems and Exercises

1. Is the atlas of the sphere exhibited in Problem 3 c) of Sect. 12.1 an orienting atlas of the sphere?

2. a) Using Example 4 of Sect. 12.1, exhibit an orienting atlas of the two-dimensional torus.

b) Prove that there does not exist an orienting atlas for the Möbius band.

c) Show that under a diffeomorphism $f : D \to \widetilde{D}$ an orientable surface $S \subset D$ maps to an orientable surface $\widetilde{S} \subset \widetilde{D}$.

3. a) Verify that the curvilinear coordinate systems on a domain $G \subset \mathbb{R}^n$ belonging to the same orientation class generate continuous fields of frames in G that determine frames of the same orientation class on the space TG_x at each point $x \in G$.

b) Show that in a connected domain $G \subset \mathbb{R}^n$ the continuous fields of frames divide into exactly two orientation classes.

c) Use the example of the sphere to show that a smooth surface $S \subset \mathbb{R}^n$ may be orientable even those there is no continuous field of frames in the tangent spaces to S.

d) Prove that on a connected orientable surface one can define exactly two different orientations.

4. a) A subspace $\mathbb{R}^{n-1}$ has been fixed, a vector $\mathrm{v} \in \mathbb{R}^n \setminus \mathbb{R}^{n-1}$ has been chosen, along with two frames $(\xi_1, \dots, \xi_{n-1})$ and $(\widetilde{\xi}_1, \dots, \widetilde{\xi}_{n-1})$ of the subspace $\mathbb{R}^{n-1}$. Verify that these frames belong to the same orientation class of frames of $\mathbb{R}^{n-1}$ if and only if the frames $(\mathrm{v}, \xi_1, \dots, \xi_{n-1})$ and $(\mathrm{v}, \widetilde{\xi}_1, \dots, \widetilde{\xi}_{n-1})$ define the same orientation on $\mathbb{R}^n$.

b) Show that a smooth hypersurface $S \subset \mathbb{R}^n$ is orientable if and only if there exists a continuous field of unit normal vectors to S. Hence, in particular, it follows that a two-sided surface is orientable.

c) Show that if $\operatorname{grad} F \neq 0$, then the surface defined by $F(x^1, \dots, x^m) = 0$ is orientable (assuming that the equation has solutions).

d) Generalize the preceding result to the case of a surface defined by a system of equations.

e) Explain why not every smooth two-dimensional surface in $\mathbb{R}^3$ can be defined by an equation $F(x, y, z) = 0$, where F is a smooth surface having no critical points (a surface for which $\operatorname{grad} F \neq 0$ at all points).

12.3 The Boundary of a Surface and its Orientation

12.3.1 Surfaces with Boundary

Let $\mathbb{R}^n$ be a Euclidean space of dimension k endowed with Cartesian coordinates $t^1, \dots, t^k$. Consider the half-space $H^k := \{t \in \mathbb{R}^k | t^1 \leq 0$ of the space $\mathbb{R}^k$. The hyperplane $\partial H^k := \{t \in \mathbb{R}^k | t^1 = 0\}$ will be called the *boundary* of the half-space H^k.

We remark that the set $\overset{o}{H}{}^k := H^k \setminus \partial H^k$, that is, the open part of H^k, is an elementary k-dimensional surface. The half-space H^k itself does not formally satisfy the definition of a surface because of the presence of the boundary points from ∂H^k. The set H^k is the standard model for surfaces with boundary, which we shall now describe.

Definition 1. A set $S \subset \mathbb{R}^n$ is a (k-dimensional) *surface with boundary* if every point $x \in S$ has a neighborhood U in S homemorphic either to $\mathbb{R}^k$ or to H^k.

Definition 2. If a point $x \in U$ corresponds to a point of the boundary ∂H^k under the homeomorphism of Definition 1, then x is called a *boundary point* of the surface (with boundary) S and of its neighborhood U. The set of all such boundary points is called the *boundary of the surface S*.

As a rule, the boundary of a surface S will be denoted ∂S. We note that for $k = 1$ the space ∂H^k consists of a single point. Hence, preserving the relation $\partial H^k = \mathbb{R}^{k-1}$, we shall from now on take $\mathbb{R}^0$ to consist of a single point and regard $\partial \mathbb{R}^0$ as the empty set.

We recall that under a homeomorphism $\varphi_{ij} : G_i \to G_j$ of the domain $G_i \subset \mathbb{R}^k$ onto the domain $G_j \subset \mathbb{R}^k$ the interior points of G_i map to interior points of the image $\varphi_{ij}(G_i)$ (this a theorem of Brouwer). Consequently, the concept of a boundary point of the surface is independent of the choice of the local chart, that is, the concept is well defined.

Formally Definition 1 includes the case of the surface described in Definition 1 of Sect. 12.1. Comparing these definitions, we see that if S has no boundary points, we return to our previous definition of a surface, which can now be regarded as the definition of a surface without boundary. In this connection we note that the term "surface with boundary" is normally used when the set of boundary points is nonempty.

The concept of a smooth surface S (of class $C^{(m)}$) with boundary can be introduced, as for surfaces without boundary, by requiring that S have an atlas of charts of the given smoothness class. When doing this we assume that for charts of the form $\varphi : H^k \to U$ the partial derivatives of φ are computed at points of the boundary ∂H^k only over the domain H^k of definition of the mapping φ, that is, these derivatives are sometimes one-sided, and that the Jacobian of the mapping φ is nonzero throughout H^k.

Since $\mathbb{R}^k$ can be mapped to the cube $I^k = \{t \in \mathbb{R}^k \,|\, |t^i| < 1,\, i = 1, \dots, k\}$ by a diffeomorphism of class $C^{(\infty)}$ and in such a way that H^k maps to the portion I_H^k of the cube I^k defined by the additional condition $t^1 \leq 0$, it is clear that in the definition of a surface with boundary (even a smooth one) we could have replaced $\mathbb{R}^k$ by I^k and H^k by I_H^k or by the cube $\tilde{I}^k$ with one of its faces attached: $I^{k-1} := \{t \in \mathbb{R}^k | t^1 = 1,\, |t^i| < 1,\, i = 2, \dots, k\}$, which is obviously a cube of dimension one less.

Taking account of this always-available freedom in the choice of canonical local charts of a surface, comparing Definitions 1 and 2 and Definition 1 of Sect. 12.1, we see that the following proposition holds.

Proposition 1. *The boundary of a k-dimensional surface of class $C^{(m)}$ is itself a surface of the same smoothness class, and is a surface without boundary having dimension one less than the dimension of the original surface with boundary.*

Proof. Indeed, if $A(S) = \{(H^k, \varphi_i, U_i)\} \cup \{(\mathbb{R}^k, \varphi_j, U_j)\}$ is an atlas for the surface S with boundary, then $A(\partial S) = \{(\mathbb{R}^{k-1}, \varphi_i|_{\partial H^k = \mathbb{R}^{k-1}}, \partial U_i)\}$ is obviously an atlas of the same smoothness class for ∂S. $\square$

We now give some simple examples of surfaces with boundary.

Example 1. A closed n-dimensional ball $\overline{B}^n$ in $\mathbb{R}^n$ is an n-dimensional surface with boundary. Its boundary $\partial \overline{B}^n$ is the $(n-1)$-dimensional sphere (see Figs. 12.8 and 12.9,a). The ball $\overline{B}^n$, which is often called in analogy with the two-dimensional case an *n-dimensional disk*, can be homeomorphically mapped to half of an n-dimensional sphere whose boundary is the equatorial $(n-1)$-dimensional sphere (Fig. 12.9,b).

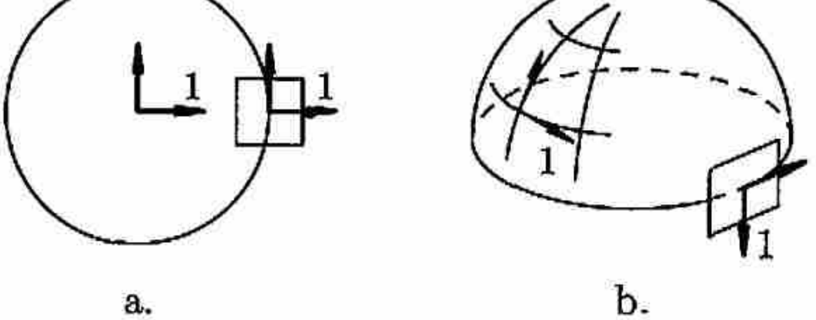


Fig. 12.9.

Example 2. The closed cube $\bar{I}^n$ in $\mathbb{R}^n$ can be homeomorphically mapped to the closed ball $\partial\overline{B}^n$ along rays emanating from its center. Consequently $\bar{I}^n$, like $\overline{B}^n$ is an n-dimensional surface with boundary, which in this case is formed by the faces of the cube (Fig. 12.10). We note that on the edges, which are the intersections of the faces, it is obvious that no mapping of the cube onto the ball can be regular (that is, smooth and of rank n).

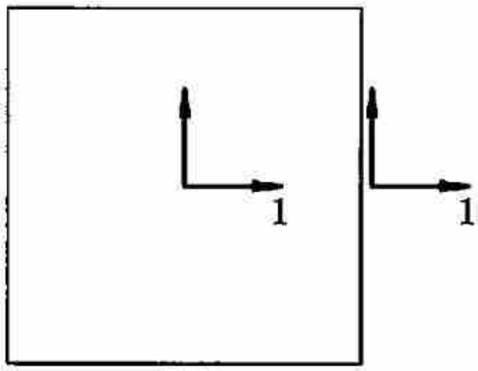
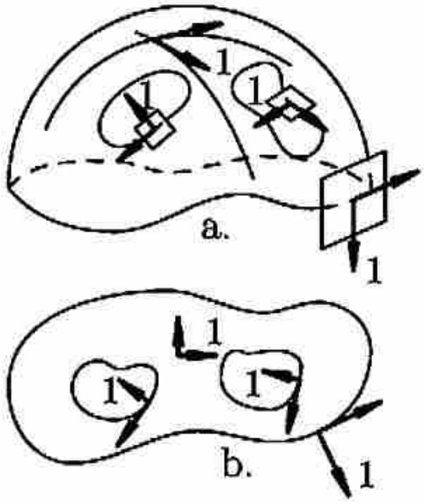


Fig. 12.10.

Example 3. If the Möbius band is obtained by gluing together two opposite sides of a closed rectangle, as described in Example 5 of Sect. 12.1, the result is obviously a surface with boundary in $\mathbb{R}^3$, and the boundary is homeomorphic to a circle (to be sure, the circle is knotted in $\mathbb{R}^3$).

Under the other possible gluing of these sides the result is a cylindrical surface whose boundary consists of two circles. This surface is homeomorphic to the usual planar annulus (see Fig. 12.3 and Example 5 of Sect. 12.1). Figures 12.11a, 12.11b, 12.12a, 12.12b, 12.13a, and 12.13b, which we will use below, show pairwise homeomorphic surfaces with boundary embedded in $\mathbb{R}^2$ and $\mathbb{R}^3$. As one can see, the boundary of a surface may be disconnected, even when the surface itself is connected.



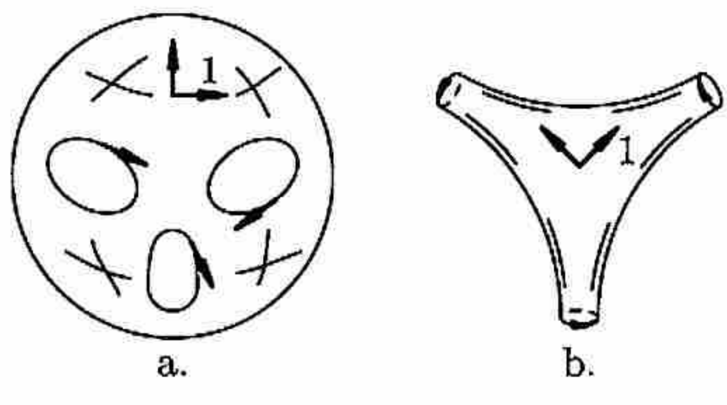


Fig. 12.12.

Fig. 12.11.

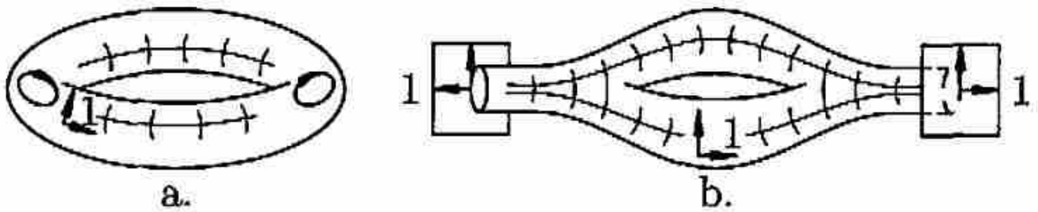


Fig. 12.13.

12.3.2 Making the Orientations of a Surface and its Boundary Consistent

If an orienting orthoframe $\mathbf{e}_1, \dots, \mathbf{e}_k$ that induces Cartesian coordinates $x^1, \dots, x^k$ is fixed in $\mathbb{R}^k$, the vectors $\mathbf{e}_2, \dots, \mathbf{e}_k$ define an orientation on the boundary $\partial H^k = \mathbb{R}^{k-1}$ of $\partial H^k = \{x \in \mathbb{R}^k | x^1 \le 0\}$ which is regarded as the orientation of the half-space H^k consistent with the orientation of the half-space H^k given by the frame $\mathbf{e}_1, \dots, \mathbf{e}_k$.

In the case $k = 1$ where $\partial H^k = \mathbb{R}^{k-1} = \mathbb{R}^0$ is a point, a special convention needs to be made as to how to orient the point. By definition, the point is oriented by assigning a sign $+$ or $-$ to it. In the case $\partial H^1 = \mathbb{R}^0$, we take $(\mathbb{R}^0, +)$, or more briefly $+\mathbb{R}^0$.

We now wish to determine what is meant in general by consistency of the orientation of a surface and its boundary. This is very important in carrying out computations connected with surface integrals, which will be discussed below.

We begin by verifying the following general proposition.

Proposition 2. *The boundary ∂S of a smooth orientable surface S is itself a smooth orientable surface (although possibly not connected).*

Proof. After we take account of Proposition 1 all that remains is to verify that ∂S is orientable. We shall show that if $A(S) = \{H^k, \varphi_i, U_i) \cup \{(\mathbb{R}^k, \varphi_j, U_j)\}$ is an orienting atlas for a surface S with boundary, then the atlas $A(\partial S) = \{\mathbb{R}^{k-1}, \varphi_i\big|_{\partial H^k = \mathbb{R}^{k-1}}, \partial U_i)\}$ of the boundary also consists of pairwise consistent charts. To do this it obviously suffices to verify that if $\tilde{t} = \psi(t)$ is a diffeomorphism with positive Jacobian from an H^k-neighborhood $U_{H^k}(t_0)$ of the point t_0 in ∂H^k onto an H^k-neighborhood $\widetilde{U}_{H^k}(\widetilde{t}_0)$ of the point $\widetilde{t}_0 \in \partial H^k$, then the mapping $\psi\big|_{\partial U_{H^k}(t_0)}$ from the H^k-neighborhood $U_{\partial H^k}(t_0) = \partial U_{H^k}(t_0)$ of $t_0 \in \partial H^k$ onto the H^k-neighborhood $\widetilde{U}_{\partial H^k}(\widetilde{t}_0) = \partial \widetilde{U}_{H^k}(\widetilde{t}_0)$ of $\widetilde{t}_0 = \psi(t_0) \in \partial H^k$ also has a positive Jacobian.

We remark that at each point $t_0 = (0, t_0^2, \dots, t_0^k) \in \partial H^k$ the Jacobian J of the mapping ψ has the form

$$J(t_0) = \begin{vmatrix} \frac{\partial \psi^1}{\partial t^1} & 0 & \cdots & 0 \\ \frac{\partial \psi^2}{\partial t^1} & \frac{\partial \psi^2}{\partial t^2} & \cdots & \frac{\partial \psi^2}{\partial t^k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi^k}{\partial t^1} & \frac{\partial \psi^k}{\partial t^2} & \cdots & \frac{\partial \psi^k}{\partial t^k} \end{vmatrix} = \frac{\partial \psi^1}{\partial t^1} \begin{vmatrix} \frac{\partial \psi^2}{\partial t^2} & \cdots & \frac{\partial \psi^2}{\partial t^k} \\ \cdots & \cdots & \cdots \\ \frac{\partial \psi^k}{\partial t^2} & \cdots & \frac{\partial \psi^k}{\partial t^k} \end{vmatrix},$$

since for $t^1 = 0$ we must also have $\widetilde{t}^1 = \psi^1(0, t^2, \dots, t^k) \equiv 0$ (boundary points map to boundary points under a diffeomorphism). It now remains only to remark that when $t^1 < 0$ we must also have $\widetilde{t}^1 = \psi^1(t^1, t^2, \dots, t^k) < 0$ (since $\widetilde{t} = \psi(t) \in H^k$), so that the value of $\frac{\partial \psi^1}{\partial t^1}(0, t^2, \dots, t^k)$ cannot be negative. By hypothesis $J(t_0) > 0$, and since $\frac{\partial \psi^1}{\partial t^1}(0, t^2, \dots, t^k) > 0$ it follows from the equality given above connecting the determinants that the Jacobian of the mapping $\psi\big|_{\partial U_{H^k}} = \psi(0, t^2, \dots, t^k)$ is positive. □

We note that the case of a one-dimensional surface ($k = 1$) in Proposition 2 and Definition 3 below must be handled by a special convention in accordance with the convention adopted at the beginning of this subsection.

Definition 3. If $A(S) = \{(H^k, \varphi_i, U_i)\} \cap \{(\mathbb{R}^k, \varphi_j, U_j)\}$ is an orienting atlas of standard local charts of the surface S with boundary ∂S, then $A(\partial S) = \{(\mathbb{R}^{k-1}, \varphi\big|_{\partial H^k = \mathbb{R}^{k-1}}, \partial U_i)\}$ is an orienting atlas for the boundary. The orientation of ∂S that it defines is said to be the orientation *consistent with the orientation of the surface.*

To finish our discussion of orientation of the boundary of an orientable surface, we make two useful remarks.

Remark 1. In practice, as already noted above, an orientation of a surface embedded in $\mathbb{R}^n$ is often defined by a frame of tangent vectors to the surface. For that reason, the verification of the consistency of the orientation of the surface and its boundary in this case can be carried out as follows. Take a k-dimensional plane TS_{x_0} tangent to the smooth surface S at the point $x_0 \in \partial S$. Since the local structure of S near x_0 is the same as the structure of the half-space H^k near $0 \in \partial H^k$, directing the first vector of the orthoframe $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_k \in TS_{x_0}$ along the normal to ∂S and in the direction exterior to the local projection of S on TS_{x_0}, we obtain a frame $\boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_k$ in the $(k-1)$-dimensional plane $T\partial S_{x_0}$ tangent to ∂S at x_0, which defines an orientation of $T\partial S_{x_0}$, and hence also of ∂S, consistent with orientation of the surface S defined by the given frame $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_k$.

Figures 12.9–12.12 show the process and the result of making the orientations of a surface and its boundary consistent using a simple example.

We note that this scheme presumes that it is possible to translate a frame that defines the orientation of S to different points of the surface and its boundary, which, as examples show, may be disconnected.

Remark 2. In the oriented space $\mathbb{R}^k$ we consider the half-space $H^k_- = H^k = \{x \in \mathbb{R}^k | x^1 \le 0\}$ and $H^k_+ = \{x \in \mathbb{R}^k | x^1 \ge 0\}$ with the orientation induced from $\mathbb{R}^k$. The hyperplane $\Gamma = \{x \in \mathbb{R}^k | x^1 = 0\}$ is the common boundary of H^k_- and H^k_+. It is easy to see that the orientations of the hyperplane Γ consistent with the orientations of H^k_- and H^k_+ are opposite to each other. This also applies to the case $k = 1$, by convention.

Similarly, if an oriented k-dimensional surface is cut by some $(k-1)$-dimensional surface (for example, a sphere intersected by its equator), two opposite orientations arise on the intersection, induced by the parts of the original surface adjacent to it.

This observation is often used in the theory of surface integrals.

In addition, it can be used to determine the orientability of a piecewise-smooth surface.

We begin by giving the definition of such a surface.

Definition 4. (Inductive definition of a piecewise-smooth surface). We agree to call a point a *zero-dimensional* surface of any smoothness class.

A *piecewise smooth one-dimensional surface* (piecewise smooth curve) is a curve in $\mathbb{R}^n$ which breaks into smooth one-dimensional surfaces (curves) when a finite or countable number of zero-dimensional surfaces are removed from it.

A surface $S \subset \mathbb{R}^n$ of dimension k is *piecewise smooth* if a finite or countable number of piecewise smooth surfaces of dimension at most $k-1$ can be removed from it in such a way that the remainder decomposes into smooth k-dimensional surfaces S_i (with boundary or without).

Example 4. The boundary of a plane angle and the boundary of a square are piecewise-smooth curves.

The boundary of a cube or the boundary of a right circular cone in $\mathbb{R}^3$ are two-dimensional piecewise-smooth surfaces.

Let us now return to the orientation of a piecewise-smooth surface.

A point (zero-dimensional surface), as already pointed out, is by convention oriented by ascribing the sign + or − to it. In particular, the boundary of a closed interval $[a, b] \subset \mathbb{R}$, which consists of the two points a and b is by convention consistent with the orientation of the closed interval from a to b if the orientation is $(a, -)$, $(b, +)$, or, in another notation, $-a, +b$.

Now let us consider a k-dimensional piecewise smooth surface $S \subset \mathbb{R}^n$ $(k > 0)$.

We assume that the two smooth surfaces S_{i_1} and S_{i_2} in Definition 4 are oriented and abut each other along a smooth portion Γ of a $(k-1)$-dimensional surface (edge). Orientations then arise on Γ, which is a boundary, consistent with the orientations of S_{i_1} and S_{i_2}. If these two orientations are opposite on every edge $\Gamma \subset \overline{S}_{i_1} \cap \overline{S}_{i_2}$, the original orientations of S_{i_1} and S_{i_2} are considered *consistent*. If $\overline{S}_{i_1} \cap \overline{S}_{i_2}$ is empty or has dimension less than $(k-1)$, all orientations of S_{i_1} and S_{i_2} are consistent.

Definition 5. A piecewise-smooth k-dimensional surface ($k > 0$) will be considered *orientable* if up to a finite or countable number of piecewise-smooth surfaces of dimension at most $(k - 1)$ it is the union of smooth orientable surfaces S_i any two of which have a mutually consistent orientation.

Example 5. The surface of a three-dimensional cube, as one can easily verify, is an orientable piecewise-smooth surface. In general, all the piecewise-smooth surfaces exhibited in Example 4 are orientable.

Example 6. The Möbius band can easily be represented as the union of two orientable smooth surfaces that abut along a piece of the boundary. But these surfaces cannot be oriented consistently. One can verify that the Möbius band is not an orientable surface, even from the point of view of Definition 5.

12.3.3 Problems and Exercises

1. a) Is it true that the boundary of a surface $S \subset \mathbb{R}^n$ is the set $\overline{S} \setminus S$, where $\overline{S}$ is the closure of S in $\mathbb{R}^n$?

b) Do the surfaces $S_1 = \{(x,y) \in \mathbb{R}^2 | 1 < x^2 + y^2 < 2\}$ and $S_2 = \{(x,y) | 0 < x^2 + y^2$ have a boundary?

c) Give the boundary of the surfaces $S_1 = \{(x,y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 < 2\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2\}$.

2. Give an example of a nonorientable surface with an orientable boundary.

3. a) Each face $I^k = \{x \in \mathbb{R}^k | \, |x^i| < 1, \, i = 1, \dots, k\}$ is parallel to the corresponding $(k-1)$-dimensional coordinate hyperplane in $\mathbb{R}^k$, so that one may consider the same frame and the same coordinate system in the face as in the hyperplane. On which faces is the resulting orientation consistent with the orientation of the cube I^k induced by the orientation of $\mathbb{R}^k$, and on which is it not consistent? Consider successively the cases $k = 2$, $k = 3$, and $k = n$.

b) The local chart $(t^1, t^2) \mapsto (\sin t^2 \cos t^2, \sin t^2 \sin t^2, \cos t^1)$ acts in a certain domain of the hemisphere $S = \{(x, y, z) \in \mathbb{R}^3 | \, x^2 + y^2 + z^2 = 1 \wedge z > 0\}$, and the local chart $t \mapsto (\cos t, \sin t, 0)$ acts in a certain domain of the boundary ∂S of this hemisphere. Determine whether these charts give a consistent orientation of the surface S and its boundary ∂S.

c) Construct the field of frames on the hemisphere S and its boundary ∂S induced by the local charts shown in b).

d) On the boundary ∂S of the hemisphere S exhibit a frame that defines the orientation of the boundary consistent with the orientation of the hemisphere given in c).

e) Define the orientation of the hemisphere S obtained in c) using a normal vector to $S \subset \mathbb{R}^3$.

4. a) Verify that the Möbius band is not an orientable surface even from the point of view of Definition 5.

b) Show that if S is a smooth surface in $\mathbb{R}^n$, determining its orientability as a smooth surface and as a piecewise-smooth surface are equivalent processes.

5. a) We shall say that a set $S \subset \mathbb{R}^n$ is a k-dimensional surface with boundary if for each point $x \in S$ there exists a neighborhood $U(x) \in \mathbb{R}^n$ and a diffeomorphism $\psi : U(x) \to I^n$ of this neighborhood onto the standard cube $I^n \subset \mathbb{R}^n$ under which $\psi(S \cap U(x))$ coincides either with the cube $I^k = \{t \in I^n | t^{k+1} = \cdots = t^n = 0\}$ or with a portion of it $I^k \cap \{t \in \mathbb{R}^n | t^k \leq 0\}$ that is a k-dimensional open interval with one of its faces attached.

Based on what was said in Sect. 12.1 in the discussion of the concept of a surface, show that this definition of a surface with boundary is equivalent to Definition 1.

b) Is it true that if $f \in C^{(l)}(H^k, \mathbb{R})$, where $H^k = \{x \in \mathbb{R}^k | x^1 \leq 0\}$, then for every point $x \in \partial H^k$ one can find a neighborhood of it $U(x)$ in $\mathbb{R}^k$ and a function $\mathcal{F} \in C^{(l)}(U(x), \mathbb{R})$ such that $\mathcal{F}\Big|_{H^k \cap U(x)} = f\Big|_{H^k \cap U(x)}$?

c) If the definition given in part a) is used to describe a smooth surface with boundary, that is, we regard ψ as a smooth mapping of maximal rank, will this definition of a smooth surface with boundary be the same as the one adopted in Sect. 12.3?

12.4 The Area of a Surface in Euclidean Space

We now turn to the problem of defining the area of a k-dimensional piecewise-smooth surface embedded in the Euclidean space $\mathbb{R}^n$, $n \geq k$.

We begin by recalling that if $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$ are k vectors in Euclidean space $\mathbb{R}^k$, then the volume $V(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k)$ of the parallelepiped spanned by these vectors as edges can be computed as the determinant

$$V(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k) = \det(\xi_i^j) \tag{12.6}$$

of the matrix $J = (\xi_i^j)$ whose rows are formed by the coordinates of these vectors in some orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$ of $\mathbb{R}^k$. We note, however, that in actual fact formula (12.6) gives the so-called *oriented volume of the parallelepiped*, rather than simply the volume. If $V \neq 0$, the value of V given by (12.6) is positive or negative according as the frames $\mathbf{e}_1, \ldots, \mathbf{e}_k$ and $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$ belong to the same or opposite orientation classes of $\mathbb{R}^k$.

We now remark that the product JJ^* of the matrix J and its transpose J^* has elements that are none other than the matrix $G = (g_{ij})$ of pairwise inner products $g_{ij} = \langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_j \rangle$ of these vectors, that is, the *Gram matrix*[8] of the system of vectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$. Thus

$$\det G = \det(JJ^*) = \det J \det J^* = (\det J)^2 , \tag{12.7}$$

and hence the nonnegative value of the volume $V(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k)$ can be obtained as

$$V(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k) = \sqrt{\det\left(\langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_j \rangle\right)} . \tag{12.8}$$

[8] See the footnote on p. 503.

This last formula is convenient in that it is essentially coordinate-free, containing only a set of geometric quantities that characterize the parallelepiped under consideration. In particular, if these same vectors $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$ are regarded as embedded in n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq k$), formula (12.8) for the k-dimensional volume (or k-dimensional surface area) of the parallelepiped they span remains unchanged.

Now let $\mathbf{r} : D \to S \subset \mathbb{R}^n$ be a k-dimensional smooth surface S in the Euclidean space $\mathbb{R}^n$ defined in parametric form $\mathbf{r} = \mathbf{r}(t^1, \dots, t^k)$, that is, as a smooth vector-valued function $\mathbf{r}(t) = (x^1, \dots, x^n)(t)$ defined in the domain $D \subset \mathbb{R}^k$. Let $\mathbf{e}_1, \dots, \mathbf{e}_k$ be the orthonormal basis in $\mathbb{R}^k$ that generates the coordinate system $(t^1, \dots, t^k)$. After fixing a point $t_0 = (t_0^1, \dots, t_0^k) \in D$, we take the positive numbers $h^1, \dots, h^k$ to be so small that the parallelepiped I spanned by the vectors $h^i\mathbf{e}_i \in TD_{t_0}$, $i = 1, \dots, k$, attached at the point t_0 is contained in D.

Under the mapping $D \to S$ a figure I_S on the surface S, which we may provisionally call a curvilinear parallelepiped, corresponds to the parallelepiped I (see Fig. 12.14, which corresponds to the case $k = 2$, $n = 3$). Since

$$\mathbf{r}(t_0^1, \dots, t_0^{i-1}, t_0^i + h^i, t_0^{i+1}, \dots, t_0^k) - \\ - \mathbf{r}(t_0^1, \dots, t_0^{i-1}, t_0^i, t_0^{i+1}, \dots, t_0^k) = \frac{\partial \mathbf{r}}{\partial t^i}(t_0)h^i + o(h^i)\,,$$

a displacement in $\mathbb{R}^n$ from $\mathbf{r}(t_0)$ that can be replaced, up to $o(h^i)$, by the partial differential $\frac{\mathbf{r}}{\partial t^i}(t_0)h^i =: \dot{\mathbf{r}}_i h^i$ as $h^i \to 0$ corresponds to displacement from t_0 by $h^i\mathbf{e}_i$. Thus, for small values of h^i, $i = 1, \dots, k$, the curvilinear parallelepiped I_S differs only slightly from the parallelepiped spanned by the vectors $h^1\dot{\mathbf{r}}_1, \dots, h^i\dot{\mathbf{r}}_k$ tangent to the surface S at $\mathbf{r}(t_0)$. Assuming on that basis that the volume ΔV of the curvilinear parallelepiped I_S must also be close to the volume of the standard parallelepiped just exhibited, we find the

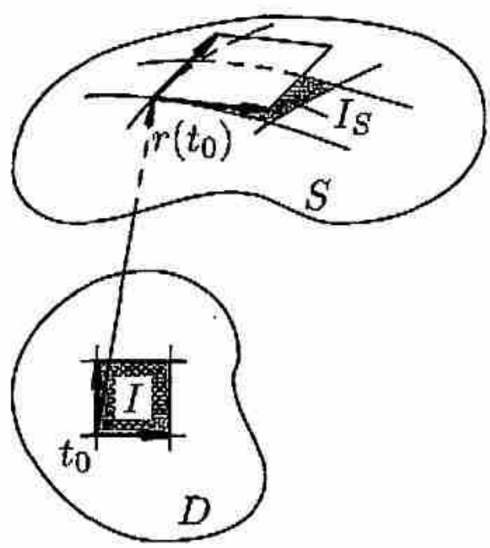


Fig. 12.14.

approximate formula

$$\Delta V \approx \sqrt{\det(g_{ij})(t_0)}\Delta t^1 \cdot \ldots \cdot \Delta t^k , \tag{12.9}$$

where we have set $g_{ij}(t_0) = \langle \dot{\mathbf{r}}_i, \dot{\mathbf{r}}_j \rangle (t_0)$ and $\Delta t^i = h^i$, $i, j = 1, \ldots, k$.

If we now tile the entire space $\mathbb{R}^k$ containing the parameter domain D with k-dimensional parallelepipeds of small diameter d, take the ones that are contained in D, compute an approximate value of the k-dimensional volume of their images using formula (12.9), and then sum the resulting values, we arrive at the quantity

$$\sum_\alpha \sqrt{\det(g_{ij})(t_\alpha)}\Delta t^1 \cdot \ldots \cdot \Delta t^k ,$$

which can be regarded as an approximation to the k-dimensional volume or area of the surface S under consideration, and this approximate value should become more precise as $d \to 0$. Thus we adopt the following definition.

Definition 1. The *area* (or *k-dimensional volume*) of a smooth k-dimensional surface S given parametrically by $D \ni t \mapsto \mathbf{r}(t) \in S$ and embedded in the Euclidean space $\mathbb{R}^n$ is the quantity

$$V_k(S) := \int_D \sqrt{\det\left(\langle \dot{\mathbf{r}}_i, \dot{\mathbf{r}}_j \rangle\right)}\, \mathrm{d}t^1 \cdots \mathrm{d}t^k . \tag{12.10}$$

Let us see how formula (12.10) looks in the cases that we already know about.

For $k = 1$ the domain $D \subset \mathbb{R}^1$ is an interval with certain endpoints a and b $(a < b)$ on the line $\mathbb{R}^1$, and S is a curve in $\mathbb{R}^n$ in this case. Thus for $k = 1$ formula (12.10) becoms the formula

$$V_1(S) = \int_a^b |\dot{\mathbf{r}}(t)|\, \mathrm{d}t = \int_a^b \sqrt{(\dot{x}^1)^2 + \cdots + (\dot{x}^n)^2(t)}\, \mathrm{d}t$$

for computing the length of a curve.

If $k = n$, then S is an n-dimensional domain in $\mathbb{R}^n$ diffeomorphic to D. In this case the Jacobian matrix $J = x'(t)$ of the mapping $D \ni (t^1, \ldots, t^n) = t \mapsto \mathbf{r}(t) = (x^1, \ldots, x^n)(t) \in S$ is a square matrix. Now using relation (12.7) and the formula for change of variable in a multiple integral, one can write

$$V_n(S) = \int_D \sqrt{\det G(t)}\, \mathrm{d}t = \int_D |\det x'(t)|\, \mathrm{d}t = \int_S \mathrm{d}x = V(S) ,$$

That is, as one should have expected, we have arrived at the volume of the domain S in $\mathbb{R}^n$.

We note that for $k = 2$, $n = 3$, that is, when S is a two-dimensional surface in $\mathbb{R}^3$, one often replaces the standard notation $g_{ij} = \langle \dot{\mathbf{r}}_i, \dot{\mathbf{r}}_j \rangle$ by the following: $\sigma := V_2(S)$, $E := g_{11} = \langle \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_1 \rangle$, $F := g_{12} = g_{21} = \langle \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2 \rangle$, $G := g_{22} = \langle \dot{\mathbf{r}}_2, \dot{\mathbf{r}}_2 \rangle$; and one writes u, v respectively instead of t^1, t^2. In this notation formula (12.10) assumes the form

$$\sigma = \iint\limits_D \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v \ .$$

In particular, if $u = x$, $v = y$, and the surface S is the graph of a smooth real-valued function $z = f(x, y)$ defined in a domain $D \subset \mathbb{R}^2$, then, as one can easily compute,

$$\sigma = \iint\limits_D \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, \mathrm{d}x\, \mathrm{d}y \ .$$

We now return once again to Definition 1 and make a number of remarks that will be useful later.

Remark 1. Definition 1 makes sense only when the integral on the right-hand side of (12.10) exists. It demonstrably exists, for example, if D is a Jordan-measurable domain and $\mathbf{r} \in C^{(1)}(\overline{D}, \mathbb{R}^n)$.

Remark 2. If the surface S in Definition 1 is partitioned into a finite number of surfaces $S_1, \dots, S_m$ with piecewise smooth boundaries, the same kind of partition of the domain D into domains $D_1, \dots, D_m$ corresponding to these surfaces will correspond to this partition. If the surface S had area in the sense of Eq. (12.10), then the quantities

$$V_k(S_\alpha) = \int\limits_{D_\alpha} \sqrt{\det \langle \dot{\mathbf{r}}_i, \dot{\mathbf{r}}_j \rangle (t)}\, \mathrm{d}t$$

are defined for each value of $\alpha = 1, \dots, m$.

By the additivity of the integral, it follows that

$$V_k(S) = \sum_\alpha V_k(S_\alpha) \ .$$

We have thus established that the area of a k-dimensional surface is additive in the same sense as the ordinary multiple integral.

Remark 3. This last remark allows us to exhaust the domain D when necessary, and thereby to extend the meaning of the formula (12.10), in which the integral may now be interpreted as an improper integral.

Remark 4. More importantly, the additivity of area can be used to define the area of an arbitrary smooth or even piecewise smooth surface (not necessarily given by a single chart).

Definition 2. Let S be an arbitrary piecewise smooth k-dimensional surface in $\mathbb{R}^n$. If, after a finite or countable number of piecewise smooth surfaces of dimension at most $k-1$ are removed, it breaks up into a finite or countable number of smooth parametrized surfaces $S_1, \dots, S_m, \dots$, we set

$$V_k(S) := \sum_\alpha V_k(S_\alpha) .$$

The additivity of the multiple integral makes it possible to verify that the quantity $V_k(S)$ so defined is independent of the way in which the surface S is partitioned into smooth pieces $S_1, \dots, S_m, \dots$, each of which is contained in the range of some local chart of the surface S.

We further remark that it follows easily from the definitions of smooth and piecewise smooth surfaces that the partition of S into parametrized pieces, as described in Definition 2, is always possible, and can even be done while observing the natural additional requirement that the partition be *locally finite*. The latter means that any compact set $K \subset S$ can intersect only a finite number of the surfaces $S_1, \dots, S_m, \dots$. This can be expressed more vividly in another way: every point of S must have a neighborhood that intersects at most a finite number of the sets $S_1, \dots, S_m, \dots$.

Remark 5. The basic formula (12.10) contains a system of curvilinear coordinates $t^1, \dots, t^k$. For that reason, it is natural to verify that the quantity $V_k(S)$ defined by (12.10) (and thereby also the quantity $V_k(S)$ from Definition 2) is invariant under a diffeomorphic transition $\widetilde{D} \ni (\tilde{t}^1, \dots, \tilde{t}^k) \mapsto t = (t^1, \dots, t^k) \in D$ to new curvilinear coordinates $\tilde{t}^1, \dots, \tilde{t}^k$ varying in the domain $\widetilde{D} \subset \mathbb{R}^k$.

Proof. For the verification it suffices to remark that the matrices

$$G = (g_{ij}) = \left(\left\langle \frac{\partial \mathbf{r}}{\partial t^i}, \frac{\partial \mathbf{r}}{\partial t^j} \right\rangle \right) \quad \text{and} \quad \widetilde{G} = (\tilde{g}_{ij}) = \left(\left\langle \frac{\partial \mathbf{r}}{\partial \tilde{t}^i}, \frac{\partial \mathbf{r}}{\partial \tilde{t}^j} \right\rangle \right)$$

at corresponding points of the domains D and $\widetilde{D}$ are connected by the relation $\widetilde{G} = J^* G J$, where $J = \left(\frac{\partial t^j}{\partial \tilde{t}^i}\right)$ is the Jacobian matrix of the mapping $\widetilde{D} \ni \tilde{t} \mapsto t \in D$ and J^* is the transpose of the matrix J. Thus, $\det \widetilde{G}(\tilde{t}) = \det G(t)(\det J)^2(\tilde{t})$, from which it follows that

$$\int_D \sqrt{\det G(t)}\, dt = \int_{\widetilde{D}} \sqrt{\det G\big(t(\tilde{t})\big)}\, |J(\tilde{t})|\, d\tilde{t} = \int_{\widetilde{D}} \sqrt{\det \widetilde{G}(\tilde{t})}\, d\tilde{t} \,. \quad \square$$

Thus, we have given a definition of the k-dimensional volume or area of a k-dimensional piecewise-smooth surface that is independent of the choice of coordinate system.

Remark 6. We precede the remark with a definition.

Definition 3. A set E embedded in a k-dimensional piecewise-smooth surface S is *a set of k-dimensional measure zero* or *has area zero* in the Lebesgue sense if for every $\varepsilon > 0$ it can be covered by a finite or countable system $S_1, \ldots, S_m, \ldots$ of (possibly intersecting) surfaces $S_\alpha \subset S$ such that $\sum\limits_{\alpha} V_k(S_\alpha) < \varepsilon$.

As one can see, this is a verbatim repetition of the definition of a set of measure zero in $\mathbb{R}^k$.

It is easy to see that in the parameter domain D of any local chart $\varphi : D \to S$ of a piecewise-smooth surface S the set $\varphi^{-1}(E) \subset D \subset \mathbb{R}^k$ of k-dimensional measure zero corresponds to such a set E. One can even verify that this is the characteristic property of sets $E \subset S$ of measure zero.

In the practical computation of areas and the surface integrals introduced below, it is useful to keep in mind that if a piecewise-smooth surface $\widetilde{S}$ has been obtained from a piecewise-smooth surface S by removing a set E of measure zero from S, then the areas of $\widetilde{S}$ and S are the same.

The usefulness of this remark lies in the fact that it is often easy to remove such a set of measure zero from a piecewise-smooth surface in such a way that the result is a smooth surface $\widetilde{S}$ defined by a single chart. But then the area of $\widetilde{S}$ and hence the area of S also can be computed directly by formula (12.10).

Let us consider some examples.

Example 1. The mapping $]0, 2\pi[\ni t \mapsto (R\cos t, R\sin t) \in \mathbb{R}^2$ is a chart for the arc $\widetilde{S}$ of the circle $x^2 + y^2 = R^2$ obtained by removing the single point $E = (R, 0)$ from that circle. Since E is a set of measure zero on S, we can write

$$V_1(S) = V_1(\widetilde{S}) = \int\limits_0^{2\pi} \sqrt{R^2 \sin^2 t + R^2 \cos^2 t}\, \mathrm{d}t = 2\pi R\,.$$

Example 2. In Example 4 of Sect. 12.1 we exhibited the following parametric representation of the two-dimensional torus S in $\mathbb{R}^3$:

$$\mathbf{r}(\varphi, \psi) = \big((b + a\cos\psi)\cos\varphi, (b + a\cos\psi)\sin\varphi, a\sin\psi\big)\,.$$

In the domain $D = \{(\varphi, \psi) | \, 0 < \varphi < 2\pi, \, 0 < \psi < 2\pi\}$ the mapping $(\varphi, \psi) \mapsto \mathbf{r}(\varphi, \psi)$ is a diffeomorphism. The image $\widetilde{S}$ of the domain D under this diffeomorphism differs from the torus by the set E consisting of the coordinate line $\varphi = 2\pi$ and the line $\psi = 2\pi$. the set E thus consists of one parallel of latitude and one meridian of longitude of the torus, and, as one can easily see, has measure zero. Hence the area of the torus can be found by formula (12.10) starting from this parametric representation, considered within the domain D.

Let us carry out the necessary computations:

$$\begin{aligned}
\dot{\mathbf{r}}_\varphi &= \big(-(b+a\cos\psi)\sin\varphi, (b+a\cos\psi)\cos\varphi, 0\big)\ ,\\
\dot{\mathbf{r}}_\psi &= (-a\sin\psi)\cos\varphi, -a\sin\psi\sin\varphi, a\cos\psi)\ ,\\
g_{11} &= \langle\dot{\mathbf{r}}_\varphi,\dot{\mathbf{r}}_\varphi\rangle = (b+a\cos\psi)^2\ ,\\
g_{12} = g_{21} &= \langle\dot{\mathbf{r}}_\varphi,\dot{\mathbf{r}}_\psi\rangle = 0\ ,\\
g_{22} &= \langle\dot{\mathbf{r}}_\psi,\dot{\mathbf{r}}_\psi\rangle = a^2\ ,\\
\det G &= \begin{vmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix} = a^2(b+a\cos\psi)^2\ .
\end{aligned}$$

Consequently,

$$V_2(S) = V_2(\widetilde{S}) = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{2\pi} a(b+a\cos\psi)\,\mathrm{d}\psi = 4\pi^2 ab\ .$$

In conclusion we note that the method indicated in Definition 2 can now be used to compute the areas of piecewise-smooth curves and surfaces.

12.4.1 Problems and Exercises

1. a) Let P and $\widetilde{P}$ be two hyperplanes in the Euclidean space $\mathbb{R}^n$, D a subdomain of P, and $\widetilde{D}$ the orthogonal projection of D on the hyperplane $\widetilde{P}$. Show that the $(n-1)$-dimensional areas of D and $\widetilde{D}$ are connected by the relation $\sigma(\widetilde{D}) = \sigma(D)\cos\alpha$, where α is the angle between the hyperplanes P and $\widetilde{P}$.

b) Taking account of the result of a), give the geometric meaning of the formula $\mathrm{d}\sigma = \sqrt{1+(f'_x)^2+(f'_y)^2}\,\mathrm{d}x\,\mathrm{d}y$ for the element of area of the graph of a smooth function $z = f(x,y)$ in three-dimensional Euclidean space.

c) Show that if the surface S in Euclidean space $\mathbb{R}^3$ is defined as a smooth vector-valued function $\mathbf{r} = \mathbf{r}(u,v)$ defined in a domain $D \subset \mathbb{R}^2$, then the area of the surface S can be found by the formula

$$\sigma(S) = \iint_D \big|[\mathbf{r}'_u, \mathbf{r}'_v]\big|\,\mathrm{d}u\,\mathrm{d}v\ ,$$

where $[\mathbf{r}'_u, \mathbf{r}'_v]$ is the vector product of $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$.

d) Verify that if the surface $S \subset \mathbb{R}^3$ is defined by the equation $F(x,y,z) = 0$ and the domain U of the surface S projects orthogonally in a one-to-one manner onto the domain D of the xy-plane, we have the formula

$$\sigma(U) = \iint_D \frac{\operatorname{grad} F}{|F'_z|}\,\mathrm{d}x\,\mathrm{d}y\ .$$

2. Find the area of the spherical rectangle formed by two parallels of latitude and two meridians of longitude of the sphere $S \subset \mathbb{R}^3$.

3. a) Let (r, φ, h) be cylindrical coordinates in $\mathbb{R}^3$. A smooth curve lying in the plane $\varphi = \varphi_0$ and defined there by the equation $\mathbf{r} = \mathbf{r}(s)$, where s is the arc length parameter, is revolved about the h-axis. Show that the area of the surface obtained by revolving the piece of this curve corresponding to the closed interval $[s_1, s_2]$ of variation of the parameter s can be found by the formula

$$\sigma = 2\pi \int_{s_1}^{s_2} r(s)\,\mathrm{d}s \,.$$

b) The graph of a smooth nonnegative function $y = f(x)$ defined on a closed interval $[a, b] \subset \mathbb{R}_+$ is revolved about the x-axis, then about the y-axis. In each of these cases, write the formula for the area of the corresponding surface of revolution as an integral over the closed interval $[a, b]$.

4. a) The center of a ball of radius 1 slides along a smooth closed plane curve of length L. Show that the area of the surface of the tubular body thereby formed is $2\pi \cdot 1 \cdot L$.

b) Based on the result of part a), find the area of the two-dimensional torus obtained by revolving a circle of radius a about an axis lying in the plane of the circle and lying at distance $b > a$ from its center.

5. Describe the helical surface defined in Cartesian coordinates (x, y, z) in $\mathbb{R}^3$ by the equation

$$y = x \tan \frac{z}{h} = 0\,, \quad |z| \le \frac{\pi}{2} h\,,$$

and find the area of the portion of it for which $r^2 < x^2 + y^2 \le R^2$.

6. a) Show that the area Ω_{n-1} of the unit sphere in $\mathbb{R}^n$ is $\frac{2(\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)}$, where $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,\mathrm{d}x$. (In particular, if n is even, then $\Gamma\left(\frac{n}{2}\right) = \left(\frac{n-2}{2}\right)!$, while if n is odd, $\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{\frac{n-1}{2}}} \sqrt{\pi}$.)

b) By verifying that the volume $V_n(r)$ of the ball of radius r in $\mathbb{R}^n$ is $\frac{(\sqrt{\pi})^n}{\Gamma\left(\frac{n+2}{2}\right)} r^n$, show that $\left.\frac{\mathrm{d}V_n}{\mathrm{d}r}\right|_{r=1} = \Omega_{n-1}$.

c) Find the limit as $n \to \infty$ of the ratio of the area of the hemisphere $\{x \in \mathbb{R}^n \mid |x| = 1 \wedge x^n > 0\}$ to the area of its orthogonal projection on the plane $x^n = 0$.

d) Show that as $n \to \infty$, the majority of the volume of the n-dimensional ball is concentrated in an arbitrarily small neighborhood of the boundary sphere, and the majority of the area of the sphere is concentrated in an arbitrarily small neighborhood of its equator.

e) Show that the following beautiful corollary on *concentration phenomena* follows from the observation made in d).

A regular function that is continuous on a sphere of large dimension is nearly constant on it (recall *pressure* in thermodynamics).

Specifically,

Let us consider, for example, functions satisfying a Lipschitz condition with a fixed constant. Then for any $\varepsilon > 0$ and $\delta > 0$ there exists N such that for $n > N$ and any function $f : S^n \to \mathbb{R}$ there exists a value c with the following properties: the area of the set on which the value of f differs from c by more than ε is at most δ times the area of the whole sphere.

7. a) Let $x_1, \dots, x_k$ be a system of vectors in Euclidean space $\mathbb{R}^n$, $n \geq k$. Show that the Gram determinant of this sytem can be represented as

$$\det\Big(\langle x_i, x_j\rangle\Big) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} P^2_{i_1 \cdots i_k} ,$$

where

$$P_{i_1 \cdots i_k} = \det \begin{pmatrix} x_1^{i_1} & \cdots & x_1^{i_k} \\ \cdots & \cdots & \cdots \\ x_k^{i_1} & \cdots & x_k^{i_k} \end{pmatrix} .$$

b) Explain the geometric meaning of the quantities $P_{i_1 \cdots i_k}$ from a) and state the result of a) as *the Pythagorean theorem for measures of arbitrary dimension* k, $1 \leq k \leq n$.

c) Now explain the formula

$$\sigma = \int_D \sqrt{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \det{}^2 \begin{pmatrix} \frac{\partial x^{i_1}}{\partial t^1} & \cdots & \frac{\partial x^{i_1}}{\partial t^k} \\ \cdots & \cdots & \cdots \\ \frac{\partial x^{i_k}}{\partial t^1} & \cdots & \frac{\partial x^{i_k}}{\partial t^k} \end{pmatrix}}\, \mathrm{d}t^1 \cdots \mathrm{d}t^n$$

for the area of a k-dimensional surface given in the parametric form $x = x(t^1, \dots, t^k)$, $t \in D \subset \mathbb{R}^k$.

8. a) Verify that the quantity $V_k(S)$ in Definition 2 really is independent of the method of partitioning the surface S into smooth pieces $S_1, \dots, S_m, \dots$.

b) Show that a piecewise-smooth surface S admits the locally finite partition into pieces $S_1, \dots, S_m, \dots$ described in Definition 2.

c) Show that a set of measure 0 can always be removed from a piecewise-smooth surface S so as to leave a smooth surface $\widetilde{S} = S \setminus E$ that can be described by a single standard local chart $\varphi : I \to S$.

9. The length of a curve, like the high-school definition of the circumference of a circle, is often defined as the limit of the lengths of suitably inscribed broken lines. The limit is taken as the length of the links in the inscribed broken lines tend to zero. The following simple example, due to H. Schwarz, shows that the analogous procedure in an attempt to define the area of even a very simple smooth surface in terms of the areas of polyhedral surfaces "inscribed" in it, may lead to an absurdity.

In a cylinder of radius R and height H we inscribe a polyhedron as follows. Cut the cylinder into m equal cylinders each of height H/m by means of horizontal planes. Break each of the $m+1$ circles of intersection (including the upper and lower bases of the original cylinder) into n equal parts so that the points of division on each circle lie beneath the midpoints of the points of division of the circle immediately above. We now take a pair of division points of each circle and the point lying directly above or below the midpoint of the arc whose endpoints they are.

These three points form a triangle, and the set of all such triangles forms a polyhedral surface inscribed in the original cylindrical surface (the lateral surface of a right circular cylinder). In shape this polyhedron resembles the calf of a boot that has been crumpled like an accordion. For that reason it is often called the *Schwarz boot*.

a) Show that if m and n are made to tend to infinity in such a way that the ratio n^2/m tends to zero, then the area of the polyhedral surface just constructed will increase without bound, even though the dimensions of each of its faces (each triangle) tend to zero.

b) If n and m tend to infinity in such a way that the ratio m/n^2 tends to some finite limit p, the area of the polyhedral surfaces will tend to a finite limit, which may be larger than, smaller than, or (when $p = 0$) equal to the area of the original cylindrical surface.

c) Compare the method of introducing the area of a smooth surface described here with what was just done above, and explain why the results are the same in the one-dimensional case, but in general not in the two-dimensional case. What are the conditions on the sequence of inscribed polyhedral surfaces that guaranteee that the two results will be the same?

10. *The isoperimetric inequality*

Let $V(E)$ denote the volume of a set $E \subset \mathbb{R}^n$, and $A + B$ the (vector) sum of the sets $A, B \subset \mathbb{R}^n$. (The sum in the sense of Minkowski is meant. See Problem 4 in Sect. 11.2.)

Let B be a ball of radius h. Then $A + B =: A_h$ is the h-neighborhood of the set A.

The quantity

$$\varliminf_{h\to 0} \frac{V(A_h) - V(A)}{h} =: \mu_+(\partial A)$$

is called the *Minkowski outer area of the boundary* ∂A *of* A.

a) Show that if ∂A is a smooth or sufficiently regular surface, then $\mu_+(\partial A)$ equals the usual area of the surface ∂A.

b) Using the Brunn–Minkowski inequality (Problem 4 of Sect. 11.2), obtain now the classical *isoperimetric inequality in* $\mathbb{R}^n$:

$$\mu_+(\partial A) \geq n v^{\frac{1}{n}} V^{\frac{n-1}{n}}(A) =: \mu(S_A) ;$$

here V is the volume of the unit ball in $\mathbb{R}^n$, and $\mu(S_A)$ the area of the $((n-1)$-dimensional) surface of the ball having the same volume as A.

The isoperimetric inequality means that a solid $A \subset \mathbb{R}^n$ has boundary area $\mu_+(\partial A)$ not less than that of a ball of the same volume.

12.5 Elementary Facts about Differential Forms

We now give an elementary description of the convenient mathematical machinery known as differential forms, paying particular attention here to its algorithmic potential rather than the details of the theoretical constructions, which will be discussed in Chap. 15.

12.5.1 Differential Forms: Definition and Examples

Having studied algebra, the reader is well acquainted with the concept of a linear form, and we have already made extensive use of that concept in constructing the differential calculus. In that process we encountered mostly symmetric forms. In the present subsection we will be discussing skew-symmetric (anti-symmetric) forms.

We recall that a form $L : X^k \to Y$ of degree or order k defined on ordered sets $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$ of vectors of a vector space X and assuming values in a vector space Y is *skew-symmetric* or *anti-symmetric* if the value of the form changes sign when any pair of its arguments are interchanged, that is,

$$L(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_i, \dots, \boldsymbol{\xi}_j, \dots, \boldsymbol{\xi}_k) = -L(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_j, \dots, \boldsymbol{\xi}_i, \dots, \boldsymbol{\xi}_k) .$$

In particular, if $\boldsymbol{\xi}_i = \boldsymbol{\xi}_j$ then the value of the form will be zero, regardless of the other vectors.

Example 1. The vector (cross) product $[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]$ of two vectors in $\mathbb{R}^3$ is a skew-symmetric bilinear form with values in $\mathbb{R}^3$.

Example 2. The oriented volume $V(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k)$ of the parallelepiped spanned by the vectors $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$ of $\mathbb{R}^k$, defined by Eq. (12.6) of Sect. 12.4, is a skew-symmetric real-valued k-form on $\mathbb{R}^k$.

For the time being we shall be interested only in real-valued k-forms (the case $Y = \mathbb{R}$), even though everything that will be discussed below is applicable to the more general situation, for example, when Y is the field $\mathbb{C}$ of complex numbers.

A linear combination of skew-symmetric forms of the same degree is in turn a skew-symmetric form, that is, the skew-symmetric forms of a given degree constitute a vector space.

In addition, in algebra one introduces the *exterior product* $\wedge$ of skew-symmetric forms, which assigns to an ordered pair A^p, B^q of such forms (of degrees p and q respectively) a skew-symmetric form $A^p \wedge B^q$ of degree $p+q$. This operation is

associative: $(A^p \wedge B^q) \wedge C^r = A^p \wedge (B^q \wedge C^r)$,

distributive: $(A^p + B^p) \wedge C^q = A^p \wedge C^q + B^p \wedge C^q$,

skew-commutative: $A^p \wedge B^q = (-1)^{pq} B^q \wedge A^p$.

In particular, in the case of 1-forms A and B, we have anticommutativity $A \wedge B = -B \wedge A$, for the operations, like the anticommutativity of the vector product shown in Example 1. The exterior product of forms is in fact a generalization of the vector product.

Without going into the details of the definition of the exterior product, we take as known for the time being the properties of this operation just listed and observe that in the case of the exterior product of 1-forms $L_1, \dots, L_k \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ the result $L_1 \wedge \dots \wedge L_k$ is a k-form that assumes the value

$$L_1 \wedge \dots \wedge L_k(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \begin{vmatrix} L_1(\boldsymbol{\xi}_1) & \cdots & L_k(\boldsymbol{\xi}_1) \\ \dots & \dots & \dots \\ L_1(\boldsymbol{\xi}_k) & \cdots & L_k(\boldsymbol{\xi}_k) \end{vmatrix} = \det\left(L_j(\boldsymbol{\xi}_i)\right) \quad (12.11)$$

on the set of vectors $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$.

If relation (12.11) is taken as the definition of the left-hand side, it follows from properties of determinants that in the case of linear forms A, B, and C, we do indeed have $A \wedge B = -B \wedge A$ and $(A + B) \wedge C = A \wedge C + B \wedge C$.

Let us now consider some examples that will be useful below.

Example 3. Let $\pi^i \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $i = 1, \dots, n$, be the projections. More precisely, the linear function $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is such that on each vector $\boldsymbol{\xi} = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$ it assumes the value $\pi^i(\boldsymbol{\xi}) = \xi^i$ of the projection of that vector on the corresponding coordinate axis. Then, in accordance with formula (12.11) we obtain

$$\pi^{i_1} \wedge \dots \wedge \pi^{i_k}(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \begin{vmatrix} \xi_1^{i_1} & \cdots & \xi_1^{i_k} \\ \dots & \dots & \dots \\ \xi_k^{i_1} & \cdots & \xi_k^{i_k} \end{vmatrix} . \quad (12.12)$$

Example 4. The Cartesian coordinates of the vector product $[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]$ of the vectors $\boldsymbol{\xi}_1 = (\xi_1^1, \xi_1^2, \xi_1^3)$ and $\boldsymbol{\xi}_2 = (\xi_2^1, \xi_2^2, \xi_2^3)$ in the Euclidean space $\mathbb{R}^3$, as is known, are defined by the equality

$$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] = \left(\begin{vmatrix} \xi_1^2 & \xi_1^3 \\ \xi_2^2 & \xi_2^3 \end{vmatrix}, \begin{vmatrix} \xi_1^3 & \xi_1^1 \\ \xi_2^3 & \xi_2^1 \end{vmatrix}, \begin{vmatrix} \xi_1^1 & \xi_1^2 \\ \xi_2^1 & \xi_2^2 \end{vmatrix} \right) .$$

Thus, in accordance with the result of Example 3 we can write

$$\begin{aligned} \pi^1([\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]) &= \pi^2 \wedge \pi^3(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) , \\ \pi^2([\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]) &= \pi^3 \wedge \pi^1(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) , \\ \pi^3([\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]) &= \pi^1 \wedge \pi^2(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) . \end{aligned}$$

Example 5. Let $f : D \to \mathbb{R}$ be a function that is defined in a domain $D \subset \mathbb{R}^n$ and differentiable at $x_0 \in D$. As is known, the differential $\mathrm{d}f(x_0)$ of the function at a point is a linear function defined on displacement vectors $\boldsymbol{\xi}$ from that point. More precisely, on vectors of the tangent space TD_{x_0} to D (or $\mathbb{R}^n$) at the point under consideration. We recall that if $x^1, \dots, x^n$ are the coordinates in $\mathbb{R}^n$ and $\boldsymbol{\xi} = (\xi^1, \dots, \xi^n)$, then

$$\mathrm{d}f(x_0)(\boldsymbol{\xi}) = \frac{\partial f}{\partial x^1}(x_0)\xi^1 + \cdots + \frac{\partial f}{\partial x^n}(x_0)\xi^n = D_{\boldsymbol{\xi}} f(x_0) .$$

In particular $\mathrm{d}x^i(\boldsymbol{\xi}) = \xi^i$, or, more formally, $\mathrm{d}x^i(x_0)(\boldsymbol{\xi}) = \xi^i$. If $f_1, \dots, f_k$ are real-valued functions defined in G and differentiable at the point $x_0 \in G$, then in accordance with (12.11) we obtain

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \begin{vmatrix} \mathrm{d}f_1(\boldsymbol{\xi}_1) & \cdots & \mathrm{d}f_k(\boldsymbol{\xi}_1) \\ \cdots & \cdots & \cdots \\ \mathrm{d}f_1(\boldsymbol{\xi}_k) & \cdots & \mathrm{d}f_k(\boldsymbol{\xi}_k) \end{vmatrix} \tag{12.13}$$

at the point x_0 for the set $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$ of vectors in the space TG_{x_0}; and, in particular,

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \begin{vmatrix} \xi_1^{i_1} & \cdots & \xi_1^{i_k} \\ \cdots & \cdots & \cdots \\ \xi_k^{i_1} & \cdots & \xi_k^{i_k} \end{vmatrix} . \tag{12.14}$$

In this way skew-symmetric forms of degree k defined on the space $TD_{x_0} \approx T\mathbb{R}^n_{x_0} \approx \mathbb{R}^n$ have been obtained from the linear forms $\mathrm{d}f_1, \dots, \mathrm{d}f_k$ defined on this space.

Example 6. If $f \in C^{(1)}(D, \mathbb{R})$, where D is a domain in $\mathbb{R}^n$, then the differential $\mathrm{d}f(x)$ of the functions f is defined at any point $x \in D$, and this differential, as has been stated, is a linear function $\mathrm{d}f(x) : TD_x \to T\mathbb{R}_{f(x)} \approx \mathbb{R}$ on the tangent space TD_x to D at x. In general the form $\mathrm{d}f(x) = f'(x)$ varies in passage from one point to another in D. Thus a smooth scalar-valued function $f : D \to \mathbb{R}$ generates a linear form $\mathrm{d}f(x)$ at each point, or, as we say, generates a *field* of linear *forms* in D, defined on the corresponding tangent spaces TD_x.

Definition 1. We shall say that a real-valued *differential p-form* ω is defined in the domain $D \subset \mathbb{R}^n$ if a skew-symmetric form $\omega(x) : (TD_x)^p \to \mathbb{R}$ is defined at each point $x \in D$.

The number p is usually called the *degree* or *order* of ω. In this connection the p-form ω is often denoted ω^p.

Thus, the field of the differential $\mathrm{d}f$ of a smooth function $f : D \to \mathbb{R}$ considered in Example 6 is a differential 1-form in D, and $\omega = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ is the simplest example of a differential form of degree p.

Example 7. Suppose a vector field $D \subset \mathbb{R}^n$ is defined, that is, a vector $\mathbf{F}(x)$ is attached to each point $x \in D$. When there is a Euclidean structure in $\mathbb{R}^n$ this vector field generates the following differential 1-form $\omega^1_{\mathbf{F}}$ in D.

If $\boldsymbol{\xi}$ is a vector attached to $x \in D$, that is, $\boldsymbol{\xi} \in TD_x$, we set

$$\omega^1_{\mathbf{F}}(x)(\boldsymbol{\xi}) = \langle \mathbf{F}(x), \boldsymbol{\xi} \rangle .$$

It follows from properties of the inner product that $\omega^1_{\mathbf{F}}(x) = \langle \mathbf{F}(x), \cdot \rangle$ is indeed a linear form at each point $x \in D$.

Such differential forms arise very frequently. For example, if $\mathbf{F}$ is a continuous force field in D and $\boldsymbol{\xi}$ an infinitesimal displacement vector from the point $x \in D$, the element of work corresponding to this displacement, as is known from physics, is defined precisely by the quantity $\langle \mathbf{F}(x), \boldsymbol{\xi} \rangle$.

Thus a force field $\mathbf{F}$ in a domain D of the Euclidean space $\mathbb{R}^n$ naturally generates a differential 1-form $\omega^1_{\mathbf{F}}$ in D, which it is natural to call the *work form of the field* $\mathbf{F}$ in this case.

We remark that in Euclidean space the differential $\mathrm{d}f$ of a smooth function $f : D \to \mathbb{R}$ in the domain $D \subset \mathbb{R}^n$ can also be regarded as the 1-form generated by a vector field, in this case the field $\mathbf{F} = \operatorname{grad} f$. In fact, by definition $\operatorname{grad} f$ is such that $\mathrm{d}f(x)(\boldsymbol{\xi}) = \langle \operatorname{grad} f(x), \boldsymbol{\xi} \rangle$ for every vector $\boldsymbol{\xi} \in TD_x$.

Example 8. A vector field $\mathbf{V}$ defined in a domain D of the Euclidean space $\mathbb{R}^n$ can also be regarded as a differential form $\omega^{n-1}_{\mathbf{V}}$ of degree $n-1$. If at a point $x \in D$ we take the vector field $\mathbf{V}(x)$ and $n-1$ additional vectors $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n \in TD_x$ attached to the point x, then the oriented volume of the parallelepiped spanned by the vectors $\mathbf{V}(x), \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}$, which is the determinant of the matrix whose rows are the coordinates of these vectors, will obviously be a skew-symmetric $(n-1)$-form with respect to the variables $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}$.

For $n = 3$ the form $\omega^2_{\mathbf{V}}$ is the usual scalar triple product $\big(\mathbf{V}(x), \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\big)$ of vectors, one of which $\mathbf{V}(x)$ is given, resulting in a skew-symmetric 2-form $\omega^2_{\mathbf{V}} = (\mathbf{V}, \cdot, \cdot)$.

For example, if a steady flow of a fluid is taking place in the domain D and $\mathbf{V}(x)$ is the velocity vector at the point $x \in D$, the quantity $\big(\mathbf{V}(x), \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\big)$ is the element of volume of the fluid passing through the (parallelogram) area spanned by the small vectors $\boldsymbol{\xi}_1 \in TD_x$ and $\boldsymbol{\xi}_2 \in TD_x$ in unit time. By choosing different vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, we shall obtain areas (parallelograms) of different configuration, differently situated in space, all having one vertex at x. For each such area there will be, in general, a different value $\big(\mathbf{V}(x), \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\big)$ of the form $\omega^2_{\mathbf{V}}(x)$. As has been stated, this value shows how much fluid has flowed through the surface in unit time, that is, it characterizes the flux across the chosen element of area. For that reason we often call the form $\omega^2_{\mathbf{V}}$

(and indeed its multidimensional analogue $\omega_{\mathbf{V}}^{n-1}$ the *flux form of the vector vield* $\mathbf{V}$ in D.

12.5.2 Coordinate Expression of a Differential Form

Let us now investigate the coordinate expression of skew-symmetric algebraic and differential forms and show, in particular, that every differential k-form is in a certain sense a linear combination of standard differential forms of the form (12.14).

To abbreviate the notation, we shall assume summation over the range of allowable values for indices that occur as both superscripts and subscripts (as we did earlier in similar situations).

Let L be a k-linear form in $\mathbb{R}^n$. If a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ is fixed in $\mathbb{R}^n$, then each vector $\boldsymbol{\xi} \in \mathbb{R}^n$ gets a coordinate representation $\boldsymbol{\xi} = \xi^i \mathbf{e}_i$ in that basis, and the form L acquires the coordinate expression

$$L(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = L(\xi_1^{i_1}\mathbf{e}_{i_1}, \dots, \xi_k^{i_k}\mathbf{e}_{i_k}) = L(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k})\xi_1^{i_1} \cdots \xi_k^{i_k} . \quad (12.15)$$

The numbers $a_{i_1,\dots,i_k} = L(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k})$ characterize the form L completely if the basis in which they have been obtained is known. These numbers are obviously symmetric or skew-symmetric with respect to their indices if and only if the form L possesses the corresponding type of symmetry.

In the case of a skew-symmetric form L the coordinate representation can be transformed slightly. To make the direction of that transformation clear and natural, let us consider the special case of (12.15) that occurs when L is a skew-symmetric 2-form in $\mathbb{R}^3$. Then for the vectors $\boldsymbol{\xi}_1 = \xi_1^{i_1}\mathbf{e}_{i_1}$ and $\boldsymbol{\xi}_2 = \xi_2^{i_2}\mathbf{e}_{i_2}$, where $i_1, i_2 = 1, 2, 3$, we obtain

$$\begin{aligned}
L(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) &= L(\xi_1^{i_1}\mathbf{e}_{i_1}, \xi_2^{i_2}\mathbf{e}_{i_2}) = L(\mathbf{e}_{i_1}, \mathbf{e}_{i_2})\xi_1^{i_1}\xi_2^{i_2} = \\
&= L(\mathbf{e}_1, \mathbf{e}_1)\xi_1^1\xi_2^1 + L(\mathbf{e}_1, \mathbf{e}_2)\xi_1^1\xi_2^2 + L(\mathbf{e}_1, \mathbf{e}_3)\xi_1^1\xi_2^3 + \\
&\quad + L(\mathbf{e}_2, \mathbf{e}_1)\xi_1^2\xi_2^1 + L(\mathbf{e}_2, \mathbf{e}_2)\xi_1^2\xi_2^2 + L(\mathbf{e}_2, \mathbf{e}_3)\xi_1^2\xi_2^3 + \\
&\quad + L(\mathbf{e}_3, \mathbf{e}_1)\xi_1^3\xi_2^1 + L(\mathbf{e}_3, \mathbf{e}_2)\xi_1^3\xi_2^2 + L(\mathbf{e}_3, \mathbf{e}_3)\xi_1^3\xi_2^3 = \\
&= L(\mathbf{e}_1, \mathbf{e}_2)(\xi_1^1\xi_2^2 - \xi_1^2\xi_2^1) + L(\mathbf{e}_1, \mathbf{e}_3)(\xi_1^3\xi_2^3 - \xi_1^3\xi_1^2) + \\
&\quad + L(\mathbf{e}_2, \mathbf{e}_3)(\xi_1^2\xi_2^3 - \xi_1^3\xi_2^2) = \sum_{1 \le i_1 < i_2 \le 3} L(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}) \begin{vmatrix} \xi_1^{i_1} & \xi_1^{i_2} \\ \xi_2^{i_1} & \xi_2^{i_2} \end{vmatrix} ,
\end{aligned}$$

where the summation extends over all combinations of indices i_1 and i_2 that satisfy the inequalities written under the summation sign.

Similarly in the general case we can also obtain the following representation for a skew-symmetric form L:

$$L(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \sum_{1 \le i_1 < \dots < i_k \le n} L(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k}) \begin{vmatrix} \xi_1^{i_1} & \cdots & \xi_1^{i_k} \\ \cdots & \cdots & \cdots \\ \xi_k^{i_1} & \cdots & \xi_k^{i_k} \end{vmatrix} . \quad (12.16)$$

Then, in accordance with formula (12.12) this last equality can be rewritten as

$$L(\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_k) = \sum_{1\le i_1<\dots<i_k\le i_n} L(\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_k})\pi^{i_1}\wedge\dots\wedge\pi^{i_k}(\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_k)\ .$$

Thus, any skew-symmetric form L can be represented as a linear combination

$$L = \sum_{1\le i_1<\dots<i_k\le i_n} a_{i_1\dots i_k}\pi^{i_1}\wedge\dots\wedge\pi^{i_k}\ . \tag{12.17}$$

of the k-forms $\pi^{i_1}\wedge\dots\wedge\pi^{i_k}$, which are the exterior product formed from the elementary 1-forms $\pi^1,\dots,\pi^n$ in $\mathbb{R}^n$.

Now suppose that a differential k-form ω is defined in some domain $D\subset\mathbb{R}^n$ along with a curvilinear coordinate system $x^1,\dots,x^n$. At each point $x\in D$ we fix the basis $\mathbf{e}_1(x),\dots,\mathbf{e}_n(x)$ of the space TD_x, formed from the unit vectors along the coordinate axes. (For example, if $x^1,\dots,x^n$ are Cartesian coordinates in $\mathbb{R}^n$, then $\mathbf{e}_1(x),\dots,\mathbf{e}_n(x)$ is simply the frame $\mathbf{e}_1,\dots,\mathbf{e}_n$ in $\mathbb{R}^n$ translated parallel to itself from the origin to x.) Then at each point $x\in D$ we find by formulas (12.14) and (12.16) that

$$\begin{aligned}\omega(x)(\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_k) &=\\ &= \sum_{1\le i_1<\dots<i_k\le n} \omega\big(\mathbf{e}_{i_1}(x),\dots,\mathbf{e}_{i_k}(x)\big)\,\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}(\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_k)\end{aligned}$$

or

$$\omega(x) = \sum_{1\le i_1<\dots<i_k\le n} a_{i_1\dots i_k}(x)\,\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}\ . \tag{12.18}$$

Thus, every differential k-form is a combination of the elementary k-forms $\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$ formed from the differentials of the coordinates. As a matter of fact, that is the reason for the term "differential form."

The coefficients $a_{i_1\dots i_k}(x)$ of the linear combination (12.18) generally depend on the point x, that is, they are functions defined in the domain in which the form ω^k is given.

In particular, we have long known the expansion of the differential

$$\mathrm{d}f(x) = \frac{\partial f}{\partial x^1}(x)\mathrm{d}x^1+\dots+\frac{\partial f}{\partial x^n}(x)\mathrm{d}x^n\ , \tag{12.19}$$

and, as can be seen from the equalities

$$\begin{aligned}\langle\mathbf{F},\boldsymbol{\xi}\rangle &= \langle F^{i_1}\mathbf{e}_{i_1}(x),\xi^{i_2}\mathbf{e}_{i_2}(x)\rangle =\\ &= \langle\mathbf{e}_{i_1}(x),\mathbf{e}_{i_2}(x)\rangle F^{i_1}(x)\xi^{i_2} = g_{i_1i_2}(x)F^{i_1}(x)\xi^{i_2} =\\ &= g_{i_1i_2}(x)F^{i_1}(x)\,\mathrm{d}x^{i_2}(\boldsymbol{\xi})\ ,\end{aligned}$$

the expansion

$$\omega^1_{\mathbf{F}}(x) = \langle \mathbf{F}(x), \cdot \rangle = (g_{i_1 i}(x) F^{i_1}(x))\, \mathrm{d}x^i = a_i(x)\, \mathrm{d}x^i \tag{12.20}$$

also holds. In Cartesian coordinates this expansion looks especially simple:

$$\omega^1_{\mathbf{F}}(x) = \langle \mathbf{F}(x), \cdot \rangle = \sum_{i=1}^{n} F^i(x)\, \mathrm{d}x^i \ . \tag{12.21}$$

Next, the following equality holds in $\mathbb{R}^3$:

$$\omega^2_{\mathbf{V}}(x)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \begin{vmatrix} V^1(x) & V^2(x) & V^3(x) \\ \xi_1^1 & \xi_1^2 & \xi_1^3 \\ \xi_2^1 & \xi_2^2 & \xi_2^3 \end{vmatrix} =$$

$$= V^1(x) \begin{vmatrix} \xi_1^2 & \xi_1^3 \\ \xi_2^2 & \xi_2^3 \end{vmatrix} + V^2(x) \begin{vmatrix} \xi_1^3 & \xi_1^1 \\ \xi_2^3 & \xi_2^1 \end{vmatrix} + V^3(x) \begin{vmatrix} \xi_1^1 & \xi_1^2 \\ \xi_2^1 & \xi_2^2 \end{vmatrix} ,$$

from which it follows that

$$\omega^2_{\mathbf{V}}(x) = V^1(x)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + V^2(x)\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + V^3(x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \ . \tag{12.22}$$

Similarly, expanding the determinant of order n for the form $\omega^{n-1}_{\mathbf{V}}$ by minors along the first row, we obtain the expansion

$$\omega^{n-1}_{\mathbf{V}} = \sum_{i=1}^{n-1} (-1)^{i+1} V^i(x)\, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n , \tag{12.23}$$

where the sign $\frown$ stands over the differential that is to be omitted in the indicated term.

12.5.3 The Exterior Differential of a Form

All that has been said up to now about differential forms essentially involved each individual point x of the domain of definition of the form and had a purely algebraic character. The operation of (exterior) differentiation of such forms is specific to analysis.

Let us agree from now on to define the 0-*forms* in a domain to be functions $f : D \to \mathbb{R}$ defined in that domain.

Definition 2. The (*exterior*) *differential* of a 0-form f, when f is a differentiable function, is the usual differential $\mathrm{d}f$ of that function.

If a differential p-form ($p \geq 1$) defined in a domain $D \subset \mathbb{R}^n$

$$\omega(x) = a_{i_1 \cdots i_p}(x)\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$$

has differentiable coefficients $a_{i_1 \cdots i_p}(x)$, then its (*exterior*) *differential* is the form

$$d\omega(x) = \mathrm{d}a_{i_1 \cdots i_p}(x) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \ .$$

Using the expansion (12.19) for the differential of a function, and relying on the distributivity of the exterior product of 1-forms, which follows from relation (12.11), we conclude that

$$\mathrm{d}\omega(x) = \frac{\partial a_{i_1\cdots i_p}}{\partial x^i}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = \\ = \alpha_{ii_1\cdots i_p}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} ,$$

that is, the (exterior) differential of a p-form ($p \geq 0$) is always a form of degree $p+1$.

We note that Definition 1 given above for a differential p-form in a domain $D \subset \mathbb{R}^n$, as one can now understand, is too general, since it does not in any way connect the forms $\omega(x)$ corresponding to different points of the domain D. In actuality, the only forms used in analysis are those whose coordinates $a_{i_1\cdots i_p}(x)$ in a coordinate representation are sufficiently regular (most often infinitely differentiable) functions in the domain D. The *order of smoothness of the form* ω in the domain $D \subset \mathbb{R}^n$ is customarily characterized by the smallest order of smoothness of its coefficients. The totality of all forms of degree $p \geq 0$ with coefficients of class $C^{(\infty)}(D,\mathbb{R})$ is most often denoted $\Omega^p(D,\mathbb{R})$ or Ω^p.

Thus the operation of differentiation of forms that we have defined effects a mapping $\mathrm{d} : \Omega^p \to \Omega^{p+1}$.

Let us consider several useful specific examples.

Example 9. For a 0-form $\omega = f(x,y,z)$ – a differentiable function – defined in a domain $D \subset \mathbb{R}^3$, we obtain

$$\mathrm{d}\omega = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z .$$

Example 10. Let

$$\omega(x,y) = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$$

be a differential 1-form in a domain D of $\mathbb{R}^2$ endowed with coordinates (x,y). Assuming that P and Q are differentiable in D, by Definition 2 we obtain

$$\begin{aligned}\mathrm{d}\omega(x,y) &= \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y = \\ &= \Big(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y\Big) \wedge \mathrm{d}x + \Big(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y\Big) \wedge \mathrm{d}y = \\ &= \frac{\partial P}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial Q}{\partial x}\mathrm{d}x \wedge \mathrm{d}y = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)(x,y)\,\mathrm{d}x \wedge \mathrm{d}y .\end{aligned}$$

Example 11. For a 1-form

$$\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

defined in a domain D in $\mathbb{R}^3$ we obtain

$$\mathrm{d}\omega = \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\mathrm{d}y \wedge \mathrm{d}z + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\mathrm{d}z \wedge \mathrm{d}x + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathrm{d}x \wedge \mathrm{d}y .$$

Example 12. Computing the differential of the 2-form

$$\omega = P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y \ ,$$

where P, Q, and R are differentiable in the domain $D \subset \mathbb{R}^3$, leads to the relation

$$\mathrm{d}\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \ .$$

If (x^1, x^2, x^3) are Cartesian coordinates in the Euclidean space $\mathbb{R}^3$ and $x \mapsto f(x)$, $x \mapsto \mathbf{F}(x) = (F^1, F^2, F^3)(x)$, and $x \mapsto \mathbf{V} = (V^1, V^2, V^3)(x)$ are smooth scalar and vector fields in the domain $D \subset \mathbb{R}^3$, then along with these fields, we often consider the respective vector fields

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\right) \quad \text{—the } \textit{gradient} \text{ of } f \ , \tag{12.24}$$

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F^3}{\partial x^2} - \frac{\partial F^2}{\partial x^3}, \frac{\partial F^1}{\partial x^3} - \frac{\partial F^3}{\partial x^1}, \frac{\partial F^2}{\partial x^1} - \frac{\partial F^1}{\partial x^2}\right) \quad \text{—the } \textit{curl} \text{ of } \mathbf{F} \ , \tag{12.25}$$

and the scalar field

$$\operatorname{div} \mathbf{V} = \frac{\partial V^1}{\partial x^1} + \frac{\partial V^2}{\partial x^2} = \frac{\partial V^3}{\partial x^3} \quad \text{—the } \textit{divergence} \text{ of } \mathbf{V} \ . \tag{12.26}$$

We have already mentioned the gradient of a scalar field earlier. Without dwelling on the physical content of the curl and divergence of a vector field at the moment, we note only the connections that these classical operators have with the operation of differentiating forms.

In the oriented Euclidean space $\mathbb{R}^3$ there is a one-to-one correspondence between vector fields and 1- and 2-forms:

$$\mathbf{F} \leftrightarrow \omega^1_{\mathbf{F}} = \langle \mathbf{F}, \cdot \rangle \ , \quad \mathbf{V} \leftrightarrow \omega^2_{\mathbf{V}}(\mathbf{V}, \cdot, \cdot) \ .$$

We remark also that every 3-form in the domain $D \subset \mathbb{R}^3$ has the form $\rho(x^1, x^2, x^3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. Taking this circumstance into account, one can introduce the following definitions for $\operatorname{grad} f$, $\operatorname{curl} \mathbf{F}$, and $\operatorname{div} \mathbf{V}$:

$$f \mapsto \omega^0 (= f) \mapsto d\omega^0 (= \mathrm{d}f) = \omega^1_{\mathbf{g}} \mapsto \mathbf{g} := \operatorname{grad} f \ , \tag{12.24$'$}$$

$$\mathbf{F} \mapsto \omega^1_{\mathbf{F}} \mapsto \mathrm{d}\omega^1_{\mathbf{F}} = \omega^2_{\mathbf{r}} \mapsto \mathbf{r} := \operatorname{curl} \mathbf{F} \ , \tag{12.25$'$}$$

$$\mathbf{V} \mapsto \omega^2_{\mathbf{V}} \mapsto \mathrm{d}\omega^2_{\mathbf{V}} = \omega^3_{\rho} \mapsto \rho := \operatorname{div} \mathbf{V} \ . \tag{12.26$'$}$$

Examples 9, 11, and 12 show that when we do this in Cartesian coordinates, we arrive at the expressions (12.24), (12.25), and (12.26) above for $\operatorname{grad} f$, $\operatorname{curl} \mathbf{F}$, and $\operatorname{div} \mathbf{V}$. Thus these operators in field theory can be regarded as concrete manifestations of the operation of differentiation of exterior forms, which is carried out in a single manner on forms of any degree. More details on the gradient, curl, and divergence will be given in Chap. 14.

12.5.4 Transformation of Vectors and Forms under Mappings

Let us consider in more detail what happens with functions (0-forms) under a mapping of their domains.

Let $\varphi : U \to V$ be a mapping of the domain $U \subset \mathbb{R}^m$ into the domain $V \subset \mathbb{R}^n$. Under the mapping φ each point $t \in U$ maps to a definite point $x = \varphi(t)$ of the domain V.

If a function f is defined on V, then, because of the mapping $\varphi : U \to V$ a function $\varphi^* f$ naturally arises on the domain U, defined by the relation

$$(\varphi^* f)(t) := f\bigl(\varphi(t)\bigr) ,$$

that is, to find the value of $\varphi^* f$ at a point $t \in U$ one must send t to the point $x = \varphi(t) \in V$ and compute the value of f at that point.

Thus, if the domain U maps to the domain V under the mapping $\varphi : U \to V$, then the set of functions defined on V maps (in the opposite direction) to the set of functions defined on U under the correspondence $f \mapsto \varphi^* f$ just defined.

In other words, we have shown that a mapping $\varphi^* : \Omega^0(V) \to \Omega^0(U)$ transforming 0-forms defined on V into 0-forms defined on U naturally arises from a mapping $\varphi : U \to V$.

Now let us consider the general case of transformation of forms of any degree.

Let $\varphi : U \to V$ be a smooth mapping of a domain $U \subset \mathbb{R}^m_t$ into a domain $V \subset \mathbb{R}^n_x$, and $\varphi'(t) : TU_t \to TV_{x=\varphi(t)}$ the mapping of tangent spaces corresponding to φ, and let ω be a p-form in the domain V. Then one can assign to ω the p-form $\varphi^*\omega$ in the domain U defined at $t \in U$ on the set of vectors $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_p \in TU_t$ by the equality

$$\varphi^*\omega(t)(\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_p) := \omega\bigl(\varphi(t)\bigr)\bigl(\varphi'_1\boldsymbol{\tau}_1, \dots, \varphi'_p\boldsymbol{\tau}_p\bigr) . \qquad (12.27)$$

Thus to each smooth mapping $\varphi : U \to V$ there corresponds a mapping $\omega^* : \Omega^p(V) \to \Omega^p(U)$ that transforms forms defined on V into forms defined on U. It obviously follows from (12.27) that

$$\varphi^*(\omega' + \omega'') = \varphi^*(\omega') + \varphi^*(\omega'') , \qquad (12.28)$$

$$\varphi^*(\lambda\omega) = \lambda\varphi^*\omega , \quad \text{if } \lambda \in \mathbb{R} . \qquad (12.29)$$

Recalling the rule $(\psi \circ \varphi)' = \psi' \circ \varphi'$ for differentiating the composition of the mappings $\varphi : U \to V$, $\psi : V \to W$, we conclude in addition from (12.27) that

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* \qquad (12.30)$$

(the natural reverse path: the composition of the mappings)

$$\psi^* : \Omega^p(W) \to \Omega^p(V) , \quad \varphi^* : \Omega^p(V) \to \Omega^p(U)) .$$

Now let us consider how to carry out the transformation of forms in practice.

Example 13. In the domain $V \subset \mathbb{R}^n_x$ let us take the 2-form $\omega = \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2}$. Let $x^i = x^i(t^1, \dots, t^m)$, $i = 1, \dots, n$, be the coordinate expression for the mapping $\varphi : U \to V$ of a domain $U \subset \mathbb{R}^m_t$ into V.

We wish to find the coordinate representation of the form $\varphi^*\omega$ in U. We take a point $t \in U$ and vectors $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in TU_t$. The vectors $\boldsymbol{\xi}_1 = \varphi'(t)\boldsymbol{\tau}_1$ and $\boldsymbol{\xi}_2 = \varphi'(t)\boldsymbol{\tau}_2$ correspond to them in the space $TV_{x=\varphi(t)}$. The coordinates $(\xi_1^1, \dots, \xi_1^n)$ and $(\xi_2^1, \dots, \xi_2^n)$ of these vectors can be expressed in terms of the coordinates $(\tau_1^1, \dots, \tau_1^m)$ and $(\tau_2^1, \dots, \tau_2^m)$ of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ using the Jacobian matrix via the formulas

$$\xi_1^i = \frac{\partial x^i}{\partial t^j}(t)\tau_1^j, \quad \xi_2^i = \frac{\partial x^i}{\partial t^j}(t)\tau_2^j, \quad i = 1, \dots, n .$$

(The summation on j runs from 1 to m.)

Thus,

$$\begin{aligned}
\varphi^*\omega(t)(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) &:= \omega\big(\varphi(t)\big)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \\
&= \begin{vmatrix} \xi_1^{i_1} & \xi_1^{i_2} \\ \xi_2^{i_1} & \xi_2^{i_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial x^{i_1}}{\partial t^{j_1}}\tau_1^{j_1} & \frac{\partial x^{i_2}}{\partial t^{j_2}}\tau_1^{j_2} \\ \frac{\partial x^{i_1}}{\partial t^{j_1}}\tau_2^{j_1} & \frac{\partial x^{i_2}}{\partial t^{j_2}}\tau_2^{j_2} \end{vmatrix} = \\
&= \sum_{j_1, j_2 = 1}^{m} \frac{\partial x^{i_1}}{\partial t^{j_1}} \frac{\partial x^{i_2}}{\partial t^{j_2}} \begin{vmatrix} \tau_1^{j_1} & \tau_1^{j_2} \\ \tau_2^{j_1} & \tau_2^{j_2} \end{vmatrix} = \\
&= \sum_{j_1, j_2 = 1}^{m} \frac{\partial x^{i_1}}{\partial t^{j_1}} \frac{\partial x^{i_2}}{\partial t^{j_2}} \mathrm{d}t^{j_1} \wedge \mathrm{d}t^{j_2}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = \\
&= \sum_{1 \le j_1 < j_2 \le m} \Big(\frac{\partial x^{i_1}}{\partial t^{j_1}} \frac{\partial x^{i_2}}{\partial t^{j_2}} - \frac{\partial x^{i_1}}{\partial t^{j_2}} \frac{\partial x^{i_2}}{\partial t^{j_1}} \Big) \mathrm{d}t^{j_1} \wedge \mathrm{d}t^{j_2}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = \\
&= \sum_{1 \le j_1 < j_2 \le m} \begin{vmatrix} \frac{\partial x^{i_1}}{\partial t^{j_1}} & \frac{\partial x^{i_2}}{\partial t^{j_1}} \\ \frac{\partial x^{i_1}}{\partial t^{j_2}} & \frac{\partial x^{i_2}}{\partial t^{j_2}} \end{vmatrix} (t)\, \mathrm{d}t^{j_1} \wedge \mathrm{d}t^{j_2}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) .
\end{aligned}$$

Consequently, we have shown that

$$\varphi^*(\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2}) = \sum_{1 \le i_1 < i_2 \le m} \frac{\partial(x^{i_1}, x^{i_2})}{\partial(t^{j_1}, t^{j_2})}(t)\, \mathrm{d}t^{j_1} \wedge \mathrm{d}t^{j_2} .$$

If we use properties (12.28) and (12.29) for the operation of transformation of forms[9] and repeat the reasoning of the last example, we obtain the following equality:

[9] If (12.29) is used pointwise, one can see that

$$\varphi^*\Big(a(x)\omega\Big) = a\Big(\varphi(t)\Big)\varphi^*\omega .$$

$$\varphi^*\Big(\sum_{1\le i_1<\dots<i_p\le n} a_{i_1,\dots,i_p}(x)\,\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_p}\Big) =$$
$$= \sum_{\substack{1\le i_1<\dots<i_p\le n\\ 1\le j_1<\dots<j_p\le m}} a_{i_1,\dots,i_p}\big(x(t)\big)\frac{\partial(x^{i_1},\dots,x^{i_p})}{\partial(t^{j_1},\dots,t^{j_p})}\mathrm{d}t^{j_1}\wedge\dots\wedge\mathrm{d}t^{j_p}\,. \quad (12.31)$$

We remark that if we make the formal change of variable $x = x(t)$ in the form that is the argument of φ^* on the left, express the differentials $\mathrm{d}x^1,\dots,\mathrm{d}x^n$ in terms of the differentials $\mathrm{d}t^1,\dots,\mathrm{d}t^m$, and gather like terms in the resulting expression, using the properties of the exterior product, we obtain precisely the right-hand side of Eq. (12.31).

Indeed, for each fixed choice of indices $i_1,\dots,i_p$ we have

$$a_{i_1,\dots,i_p}(x)\,\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_p} =$$
$$= a_{i_1,\dots,i_p}\big(x(t)\big)\Big(\frac{\partial x^{i_1}}{\partial t^{j_1}}\mathrm{d}t^{j_1}\Big)\wedge\dots\wedge\Big(\frac{\partial x^{i_p}}{\partial t^{j_p}}\mathrm{d}t^{j_p}\Big) =$$
$$= a_{i_1,\dots,i_p}\big(x(t)\big)\frac{\partial x^{i_1}}{\partial t^{j_1}}\cdot\ldots\cdot\frac{\partial x^{i_p}}{\partial t^{j_p}}\mathrm{d}t^{j_1}\wedge\dots\wedge\mathrm{d}t^{j_p} =$$
$$= \sum_{1\le j_1<\cdots j_p\le m} a_{i_1,\dots,i_p}\big(x(t)\big)\frac{\partial(x^{i_1},\dots,x^{i_p})}{\partial(t^{j_1},\dots,t^{j_p})}\mathrm{d}t^{j_1}\wedge\dots\wedge\mathrm{d}t^{j_p}\,.$$

Summing such equalities over all ordered sets $1\le i_1<\dots<i_p\le n$, we obtain the right-hand side of (12.31).

Thus we have proved the following propostion, of great technical importance.

Proposition. *If a differential form ω is defined in a domain $V\subset\mathbb{R}^n$ and $\varphi: U\to V$ is a smooth mapping of a domain $U\subset\mathbb{R}^m$ into V, then the coordinate expression of the form $\varphi^*\omega$ can be obtained from the coordinate expression*

$$\sum_{1\le i_1<\dots<i_p\le n} a_{i_1,\dots,i_p}(x)\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_p}$$

of the form ω by the direct change of variable $x=\varphi(t)$ (with subsequent transformations in accordance with the properties of the exterior product).

Example 14. In particular, if $m=n=p$, relation (12.31) reduces to the equality

$$\varphi^*(\mathrm{d}x^1\wedge\dots\wedge\mathrm{d}x^n) = \det\varphi'(t)\,\mathrm{d}t^1\wedge\dots\wedge\mathrm{d}t^n\,. \quad (12.32)$$

Hence, if we write $f(x)\mathrm{d}x^1\wedge\dots\wedge\mathrm{d}x^n$ in a multiple integral instead of $f(x)\,\mathrm{d}x^1\cdots\mathrm{d}x^n$, the formula

$$\int\limits_{V=\varphi(U)} f(x)\,\mathrm{d}x = \int\limits_U f\big(\varphi(t)\big)\,\det\varphi'(t)\,\mathrm{d}t$$

for change of variable in a multiple integral via an orientation-preserving diffeomorphism (that is, when $\det \varphi'(t) > 0$) could be obtained automatically by the formal substitution $x = \varphi(t)$, just as happened in the one-dimensional case, and it could be given the following form:

$$\int\limits_{\varphi(U)} \omega = \int\limits_{U} \varphi^* \omega \,. \tag{12.33}$$

We remark in conclusion that if the degree p of the form ω in the domain $V \subset \mathbb{R}^n_x$ is larger than the dimension m of the domain $U \subset \mathbb{R}^m$ that is mapped into V via $\varphi : U \to V$, then the form $\varphi^*\omega$ on U corresponding to ω is obviously zero. Thus the mapping $\varphi^* : \Omega^p(V) \to \Omega^p(U)$ is not necessarily injective in general.

On the other hand, if $\varphi : U \to V$ has a smooth inverse $\varphi^{-1} : V \to U$, then by (12.30) and the equalities $\varphi^{-1} \circ \varphi = e_U$, $\varphi \circ \varphi^{-1} = e_V$, we find that $(\varphi)^* \circ (\varphi^{-1})^* = e^*_U$ and $(\varphi^{-1})^* \circ \varphi^* = e^*_V$. And, since e^*_U and e^*_V are the identity mappings on $\Omega^p(U)$ and $\Omega^p(V)$ respectively, the mappings $\varphi^* : \Omega^p(V) \to \Omega^p(U)$ and $\left(\varphi^{-1}\right)^* : \Omega^p(U) \to \Omega^p(V)$, as one would expect, turn out to be inverses of each other. That is, in this case, the mapping $\varphi^* : \Omega^p(V) \to \Omega^p(U)$ is bijective.

We note finally that along with the properties (12.28)–(12.30) the mapping φ^* that transfers forms, as one can verify, also satisfies the relation

$$\varphi^*(\mathrm{d}\omega) = \mathrm{d}(\varphi^*\omega) \,. \tag{12.34}$$

This theoretically important equality shows in particular that the operation of differentiation of forms, which we defined in coordinate notation, is actually independent of the coordinate system in which the differentiable form ω is written. This will be discussed in more detail in Chapt. 15.

12.5.5 Forms on Surfaces

Definition 3. We say that a *differential p-form ω is defined on a smooth surface $S \subset \mathbb{R}^n$* if a p-form $\omega(x)$ is defined on the vectors of the tangent plane TS_x to S at each point $x \in S$.

Example 15. If the smooth surface S is contained in the domain $D \subset \mathbb{R}^n$ in which a form ω is defined, then, since the inclusion $TS_x \subset TD_x$ holds at each point $x \in S$, one can consider the restriction of $\omega(x)$ to TS_x. In this way a form $\omega\big|_S$ arises, which it is natural to call the *restriction of ω to S*.

As we know, a surface can be defined parametrically, either locally or globally. Let $\varphi : U \to S = \varphi(U) \subset D$ be a parametrized smooth surface in the domain D and ω a form on D. Then we can transfer the form ω to the domain U of parameters and write $\varphi^*\omega$ in coordinate form in accordance with the algorithm given above. It is clear that the form $\varphi^*\omega$ in U obtained in this way coincides with the form $\varphi^*(\omega|_S)$.

We remark that, since $\varphi'(t) : TU_t \to TS_x$ is an isomorphism between TU_t and TS_x at every point $t \in U$, we can transfer forms both from S to U and from U to S, and so just as the smooth surfaces themselves are usually defined locally or globally by parameters, the forms on them, in the final analysis, are usually defined in the parameter domains of local charts.

Example 16. Let $\omega^2_{\mathbf{V}}$ be the flux form considered in Example 8, generated by the velocity field $\mathbf{V}$ of a flow in the domain D of the oriented Euclidean space $\mathbb{R}^3$. If S is a smooth oriented surface in D, one may consider the restriction $\omega^2_{\mathbf{V}}|_S$ of the form $\omega^2_{\mathbf{V}}$ to S. The form $\omega^2_{\mathbf{V}}|_S$ so obtained characterizes the flux across each element of the surface S.

If $\varphi : I \to S$ is a local chart of the surface S, then, making the change of variable $x = \varphi(t)$ in the coordinate expression (12.22) for the form $\omega^2_{\mathbf{V}}$, we obtain the coordinate expression for the form $\varphi^*\omega^2_{\mathbf{V}} = \varphi^*(\omega^2_{\mathbf{V}}|_S)$, which is defined on the square I, in these local coordinates of the surface.

Example 17. Let $\omega^1_{\mathbf{F}}$ be the work form considered in Example 7, generated by the force field $\mathbf{F}$ acting in a domain D of Euclidean space. Let $\varphi : I \to \varphi(I) \subset D$ be a smooth path (φ is not necessarily a homeomorphism). Then, in accordance with the general principle of restriction and transfer of forms, a form $\varphi^*\omega^1_{\mathbf{F}}$ arises on the closed interval I, whose coordinate representation $a(t)\,\mathrm{d}t$ can be obtained by the change of variable $x = \varphi(t)$ in the coordinate expression (12.21) for the form $\omega^1_{\mathbf{F}}$.

12.5.6 Problems and Exercises

1. Compute the values of the differential forms ω in $\mathbb{R}^n$ given below on the indicated sets of vectors:

a) $\omega = x^2\,\mathrm{d}x^1$ on the vector $\boldsymbol{\xi} = (1,2,3) \in T\mathbb{R}_{(3,2,1)}$.

b) $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^3 + x^1 \mathrm{d}x^2 \wedge \mathrm{d}x^4$ on the ordered pair of vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in T\mathbb{R}^4_{(1,0,0,0)}$.

c) $\omega = \mathrm{d}f$, where $f = x^1 + 2x^2 + \cdots + nx^n$, and $\boldsymbol{\xi} = (1,-,1,\dots,(-1)^{n-1}) \in T\mathbb{R}^n_{(1,1,\dots,1)}$.

2. a) Verify that the form $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ is identically zero if the indices $i_1, \dots, i_k$ are not all distinct.

b) Explain why there are no nonzero skew-symmetric forms of degree $p > n$ on an n-dimensional vector space.

c) Simplify the expression for the form

$$2\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^2 + 3\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 .$$

d) Remove the parentheses and gather like terms:

$$(x^1\,\mathrm{d}x^2 + x^2\,\mathrm{d}x^1) \wedge (x^3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + x^2\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 + x^1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3) .$$

e) Write the form $df \wedge dg$, where $f = \ln\left(1 + |x|^2\right)$, $g = \sin|x|$, and $x = (x^1, x^2, x^3)$ as a linear combination of the forms $dx^{i_1} \wedge dx^{i_2}$, $1 \le i_1 < i_2 \le 3$.

f) Verify that in $\mathbb{R}^n$

$$df^1 \wedge \cdots \wedge df^n(x) = \det\left(\frac{\partial f^i}{\partial x^j}\right)(x)\, dx^1 \wedge \cdots \wedge dx^n .$$

g) Carry out all the computations and show that for $1 \le k \le n$

$$df^1 \wedge \cdots \wedge df^k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \det\begin{vmatrix} \frac{\partial f^1}{\partial x^{i_1}} & \cdots & \frac{\partial f^1}{\partial x^{i_k}} \\ \frac{\partial f^k}{\partial x^{i_1}} & \cdots & \frac{\partial f^k}{\partial x^{i_k}} \end{vmatrix} dx^{i_1} \wedge \cdots \wedge dx^{i_k} .$$

3. a) Show that a form α of even degree commutes with any form β, that is, $\alpha \wedge \beta = \beta \wedge \alpha$.

b) Let $\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$ and $\omega^n = \omega \wedge \cdots \wedge \omega$ (n factors). Verify that

$$\omega^n = n!\, dp_1 \wedge dq^1 \wedge \cdots \wedge dp_n \wedge dq^n = (-1)^{\frac{n(n-1)}{2}} dp_1 \wedge \cdots \wedge dp_n \wedge dq^1 \wedge \cdots \wedge dq^n .$$

4. a) Write the form $\omega = df$, where $f(x) = (x^1)^2 + (x^2)^2 + \cdots + (x^n)^2$, as a combination of the forms $dx^1, \ldots, dx^n$ and find the differential $d\omega$ of ω.

b) Verify that $d^2 f \equiv 0$ for any function $f \in C^{(2)}(D, \mathbb{R})$, where $d^2 = d \circ d$, and d is exterior differentiation.

c) Show that if the coefficients $a_{i_1,\ldots,i_k}$ of the form $\omega = a_{i_1,\ldots,i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ belongs to the class $C^{(2)}(D, \mathbb{R})$, then $d^2\omega \equiv 0$ in the domain D.

d) Find the exterior differential of the form $\frac{y\,dx - x\,dy}{x^2+y^2}$ in its domain of definition.

5. If the product $dx^1 \cdots dx^n$ in the multiple integral $\int_D f(x)\, dx^1 \cdots dx^n$ is interpreted as the form $dx^1 \wedge \cdots \wedge dx^n$, then, by the result of Example 14, we have the possibility of formally obtaining the integrand in the formula for change of variable in a multiple integral. Using this recommendation, carry out the following changes of variable from Cartesian coordinates:

a) to polar coordinates in $\mathbb{R}^2$,

b) to cylindrical coordinates in $\mathbb{R}^3$,

c) to spherical coordinates in $\mathbb{R}^3$.

6. Find the restriction of the following forms:

a) dx^i to the hyperplane $x^i = 1$.

b) $dx \wedge dy$ to the curve $x = x(t)$, $y = y(t)$, $a < t < b$.

c) $dx \wedge dy$ to the plane in $\mathbb{R}^3$ defined by the equation $x = c$.

d) $dy \wedge dz + dz \wedge dx + dx \wedge dy$ to the faces of the standard unit cube in $\mathbb{R}^3$.

e) $\omega_i = dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \widehat{dx^i} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$ to the faces of the standard unit cube in $\mathbb{R}^n$. The symbol $\frown$ stands over the differential dx^i that is to be omitted in the product.

7. Express the restriction of the following forms to the sphere of radius R with center at the origin in spherical coordinates on $\mathbb{R}^3$:

a) $\mathrm{d}x$,

b) $\mathrm{d}y$,

c) $\mathrm{d}y \wedge \mathrm{d}z$.

8. The mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is given in the form $(u, v) \mapsto (u \cdot v, 1) = (x, y)$. Find:

a) $\varphi^*(\mathrm{d}x)$,

b) $\varphi^*(\mathrm{d}y)$,

c) $\varphi^*(y\,\mathrm{d}x)$.

9. Verify that the exterior differential $\mathrm{d} : \Omega^p(D) \to \Omega^{p+1}(D)$ has the following properties:

a) $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$,

b) $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \mathrm{d}\omega_2$, where $\deg \omega_1$ is the degree of the form ω_1.

c) $\forall \omega \in \Omega^p \ \mathrm{d}(\mathrm{d}\omega) = 0$.

d) $\forall f \in \Omega^0 \ \mathrm{d}f = \sum\limits_{i=1}^{n} \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i$.

Show that there is only one mapping $\mathrm{d} : \Omega^p(D) \to \Omega^{p+1}(D)$ having properties a), b), c), and d).

10. Verify that the mapping $\varphi^* : \Omega^p(V) \to \Omega^p(U)$ corresponding to a mapping $\varphi : U \to V$ has the following properties:

a) $\varphi^*(\omega_1 + \omega_2) = \varphi^*\omega_1 + \varphi^*\omega_2$.

b) $\varphi^*(\omega_1 \wedge \omega_2) = \varphi^*\omega_1 \wedge \varphi^*\omega_2$.

c) $\mathrm{d}\varphi^*\omega = \varphi^*\mathrm{d}\omega$.

d) If there is a mapping $\psi : V \to W$, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

11. Show that a smooth k-dimensional surface is orientable if and only if there exists a k-form on it that is not degenerate at any point.

13 Line and Surface Integrals

13.1 The Integral of a Differential Form

13.1.1 The Original Problems, Suggestive Considerations, Examples

a. The Work of a Field Let $\mathbf{F}(x)$ be a continuous force field acting in a domain G of the Euclidean space $\mathbb{R}^n$. The displacment of a test particle in the field is accompanied by work. We ask how we can compute the work done by the field in moving a unit test particle along a given trajectory, more precisely, a smooth path $\gamma : I \to \gamma(I) \subset G$.

We have already touched on this problem when we studied the applications of the definite integral. For that reason we can merely recall the solution of the problem at this point, noting certain elements of the construction that will be useful in what follows.

It is known that in a constant field $\mathbf{F}$ the displacement by a vector $\boldsymbol{\xi}$ is associated with an amount of work $(\mathbf{F}, \boldsymbol{\xi})$.

Let $t \mapsto \mathbf{x}(t)$ be a smooth mapping $\gamma : I \to G$ defined on the closed interval $I = \{t \in \mathbb{R} | \, a \le t \le b\}$.

We take a sufficiently fine partition of the closed interval $[a, b]$. Then on each interval $I_i = \{t \in I | \, t_{i-1} \le t \le t_i\}$ of the partition we have the equality $\mathbf{x}(t) - \mathbf{x}(t_i) \approx \mathbf{x}'(t)(t_i - t_{i-1})$ up to infinitesimals of higher order. To the displacement vector $\tau_i = t_{i+1} - t_i$ from the point t_i (Fig. 13.1) there

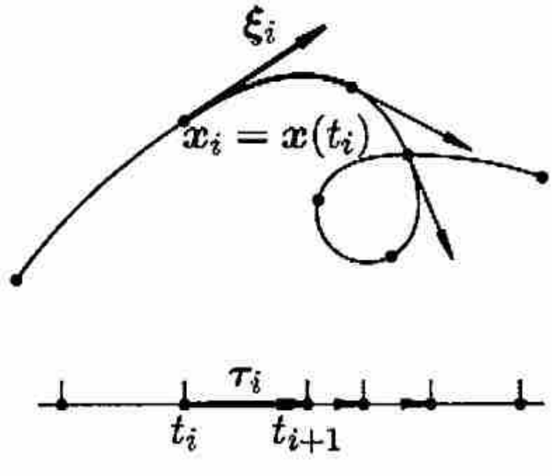


Fig. 13.1.

corresponds a displacement of $\mathbf{x}(t_i)$ in $\mathbb{R}^n$ by the vector $\Delta\mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$, which can be regarded as equal to the tangent vector $\boldsymbol{\xi}_i = \dot{\mathbf{x}}(t_i)\tau_i$ to the trajectory at $\mathbf{x}(t_i)$ with the same precision. Since the field $\mathbf{F}(\mathbf{x})$ is continuous, it can be regarded a locally constant, and for that reason we can compute the work ΔA_i corresponding to the (time) interval I_i with small relative error as

$$\Delta A_i \approx \langle \mathbf{F}(x_i), \boldsymbol{\xi}_i \rangle$$

or

$$\Delta A_i \approx \langle \mathbf{F}\bigl(\mathbf{x}(t_i)\bigr), \dot{\mathbf{x}}(t_i)\tau_i \rangle .$$

Hence,

$$A = \sum_i \Delta A_i \approx \sum_i \langle \mathbf{F}\bigl(\mathbf{x}(t_i)\bigr), \dot{\mathbf{x}}(t_i)\rangle \Delta t_i$$

and so, passing to the limit as the partition of the closed interval I is refined, we find that

$$A = \int_a^b \langle \mathbf{F}\bigl(\mathbf{x}(t)\bigr), \dot{\mathbf{x}}(t)\rangle \, \mathrm{d}t . \tag{13.1}$$

If the expression $\langle \mathbf{F}\bigl(\mathbf{x}(t)\bigr), \dot{\mathbf{x}}(t)\rangle \, \mathrm{d}t$ is rewritten as $\langle \mathbf{F}(\mathbf{x}), \mathrm{d}\mathbf{x}\rangle$, then, assuming the coordinates in $\mathbb{R}^n$ are Cartesian coordinates, we can give this expression the form $F^1 \mathrm{d}x^1 + \cdots + F^n \mathrm{d}x^n$, after which we can write (13.1) as

$$A = \int_\gamma F^1 \mathrm{d}x^1 + \cdots + F^n \mathrm{d}x^n \tag{13.2}$$

or as

$$A = \int_\gamma \omega^1_{\mathbf{F}} . \tag{13.2'}$$

Formula (13.1) provides the precise meaning of the integrals of the work 1-form along the path γ written in formulas (13.2) and (13.2′).

Example 1. Consider the force field $\mathbf{F} = \bigl(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\bigr)$ defined at all points of the plane $\mathbb{R}^2$ except the origin. Let us compute the work of this field along the curve γ_1 defined as $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, and along the curve defined by $x = 2 + \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. According to formulas (13.1), (13.2), and (13.2′), we find

$$\int_{\gamma_1} \omega^1_{\mathbf{F}} = \int_{\gamma_1} -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y =$$

$$= \int_0^{2\pi} \Bigl(-\frac{\sin t \cdot (-\sin t)}{\cos^2 t + \sin^2 t} + \frac{\cos t \cdot \cos t}{\cos^2 t + \sin^2 t} \Bigr) \mathrm{d}t = 2\pi$$

and

$$\int_{\gamma_2} \omega_{\mathbf{F}}^1 = \int_{\gamma_2} \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2+y^2} = \int_0^{2\pi} \frac{-\sin t(-\sin t) + (2+\cos t)(\cos t)}{(2+\cos t)^2 + \sin^2 t}\,\mathrm{d}t =$$

$$= \int_0^{2\pi} \frac{1+2\cos t}{5+4\cos t}\,\mathrm{d}t = \int_0^{\pi} \frac{1+2\cos t}{5+4\cos t}\,\mathrm{d}t + \int_{\pi}^{0} \frac{1+2\cos(2\pi-u)}{5+4\cos(2\pi-u)}\,\mathrm{d}u =$$

$$= \int_0^{\pi} \frac{1+2\cos t}{5+4\cos t}\,\mathrm{d}t - \int_0^{\pi} \frac{1+2\cos u}{5+4\cos u}\,\mathrm{d}u = 0.$$

Example 2. Let $\mathbf{r}$ be the radius vector of a point $(x,y,z) \in \mathbb{R}^3$ and $r = |\mathbf{r}|$. Suppose a force field $\mathbf{F} = f(r)\mathbf{r}$ is defined everywhere in $\mathbb{R}^3$ except at the origin. This is a so-called *central force field*. Let us find the work of $\mathbf{F}$ on a path $\gamma : [0,1] \to \mathbb{R}^3 \setminus 0$. Using (13.2), we find

$$\int_{\gamma} f(r)(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z) = \frac{1}{2}\int_{\gamma} f(r)\mathrm{d}(x^2+y^2+z^2) =$$

$$= \frac{1}{2}\int_0^1 f\big(r(t)\big)\mathrm{d}r^2(t) = \frac{1}{2}\int_0^1 f\big(\sqrt{u(t)}\big)\,\mathrm{d}u(t) =$$

$$= \frac{1}{2}\int_{r_0^2}^{r_1^2} f\big(\sqrt{u}\big)\,\mathrm{d}u = \Phi(r_0, r_1)\,.$$

Here, as one can see, we have set $x^2(t)+y^2(t)+z^2(t) = r^2(t)$, $r^2(t) = u(t)$, $r_0 = r(0)$, and $r_1 = r(1)$.

Thus in any central field the work on a path γ has turned out to depend only on the distances r_0 and r_1 of the beginning and end of the path from the center 0 of the field.

In particular, for the gravitational field $\frac{1}{r^3}\mathbf{r}$ of a unit point mass located at the origin, we obtain

$$\Phi(r_0, r_1) = \frac{1}{2}\int_{r_0^2}^{r_1^2} \frac{1}{u^{3/2}}\,\mathrm{d}u = \frac{1}{r_0} - \frac{1}{r_1}.$$

b. The Flux Across a Surface Suppose there is a steady flow of liquid (or gas) in a domain G of the oriented Euclidean space $\mathbb{R}^3$ and that $x \mapsto \mathbf{V}(x)$ is the velocity field of this flow. In addition, suppose that a smooth oriented surface S has been chosen in G. For definiteness we shall suppose that the orientation of S is given by a field of normal vectors. We ask how to determine the (volumetric) outflow or flux of fluid across the surface S. More precisely, we ask how to find the volume of fluid that flows across the surface S per unit time in the direction indicated by the orienting field of normals to the surface.

To solve the problem, we remark that if the velocity field of the flow is constant and equal to $\mathbf{V}$, then the flow per unit time across a parallelogram Π spanned by vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ equals the volume of the parallelepiped constructed on the vectors $\mathbf{V}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2$. If $\boldsymbol{\eta}$ is normal to Π and we seek the flux across Π in the direction of $\boldsymbol{\eta}$, it equals the scalar triple product $(\mathbf{V}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$, provided $\boldsymbol{\eta}$ and the frame $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ give Π the same orientation (that is, if $\boldsymbol{\eta}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ is a frame having the given orientation in $\mathbb{R}^3$). If the frame $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ gives the orientation opposite to the one given by $\boldsymbol{\eta}$ in Π, then the flow in the direction of $\boldsymbol{\eta}$ is $-(\mathbf{V}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$.

We now return to the original statement of the problem. For simplicity let us assume that the entire surface S admits a smooth parametrization $\varphi : I \to S \subset G$, where I is a two-dimensional interval in the plane $\mathbb{R}^2$. We partition I into small intervals I_i (Fig. 13.2). We approximate the image $\varphi(I_i)$ of each such interval by the parallelogram spanned by the images $\boldsymbol{\xi}_1 = \varphi'(t_i)\boldsymbol{\tau}_1$ and $\boldsymbol{\xi}_2 = \varphi'(t_i)\boldsymbol{\tau}_2$ of the displacement vectors $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ along the coordinate directions. Assuming that $\mathbf{V}(x)$ varies by only a small amount inside the piece of surface $\varphi(I_i)$ and replacing $\varphi(I_i)$ by this parallelogram, we may assume that the flux $\Delta\mathcal{F}_i$ across the piece $\varphi(I_i)$ of the surface is equal, with small relative error, to the flux of a constant velocity field $\mathbf{V}(x_i) = \mathbf{V}\big(\varphi(t_i)\big)$ across the parallelogram spanned by the vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$.

Assuming that the frame $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ gives the same orientation on S as $\boldsymbol{\eta}$, we find

$$\Delta\mathcal{F}_i \approx \big(\mathbf{V}(x_i), \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\big) .$$

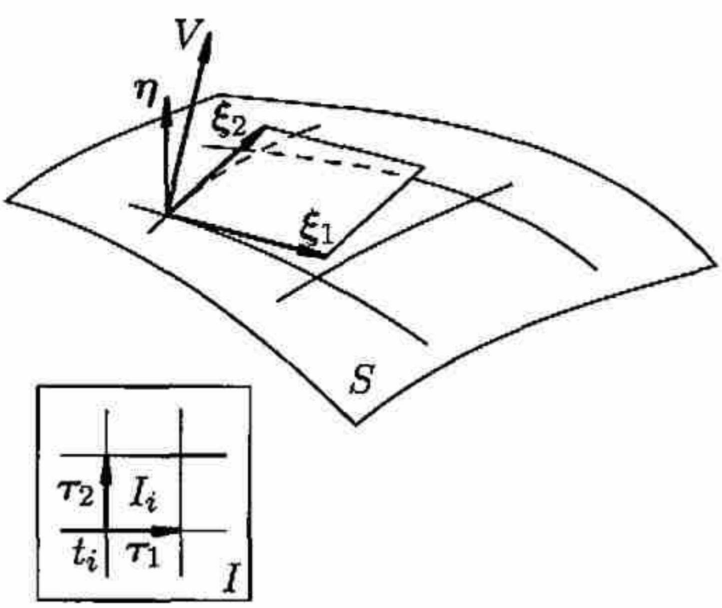


Fig. 13.2.

Summing the elementary fiuxes, we obtain

$$\mathcal{F} = \sum_i \Delta\mathcal{F}_i \approx \sum_i \omega^2_{\mathbf{V}}(x_i)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) ,$$

where $\omega^2_{\mathbf{V}}(x) = (\mathbf{V}(x), \cdot, \cdot)$ is the fiux 2-form (studied in Example 8 of Sect. 12.5). If we pass to the limit, taking ever finer partitions P of the interval I, it is natural to assume that

$$\mathcal{F} := \lim_{\lambda(P)\to 0} \sum \omega^2_{\mathbf{V}}(x_i)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) =: \int_S \omega^2_{\mathbf{V}} . \tag{13.3}$$

This last symbol is the integral of the 2-form $\omega^2_{\mathbf{V}}$ over the oriented surface S.

Recalling (formula (12.22) of Sect. 12.5) the coordinate expression for the flux form $\omega^2_{\mathbf{V}}$ in Cartesian coordinates, we may now also write

$$\mathcal{F} = \int_S V^1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + V^2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + V^3 \mathrm{d}x^1 \wedge \mathrm{d}x^2 . \tag{13.4}$$

We have dicussed here only the general principle for solving this problem. In essence all we have done is to give the precise definition (13.3) of the fiux $\mathcal{F}$ and introduced certain notation (13.3) and (13.4); we have still not obtained any effective computational formula similar to formula (13.1) for the work.

We remark that formula (13.1) can be obtained from (13.2) by replacing $x^1, \dots, x^n$ with the functions $(x^1, \dots, x^n)(t) = x(t)$ that define the path γ. We recall (Sect. 12.5) that such a substitution can be interpreted as the transfer of the form ω defined in G to the closed interval $I = [a, b]$.

In a completely analogous way, a computational formula for the fiux can be obtained by direct substitution of the parametric equations of the surface into (13.4).

In fact,

$$\omega^2_{\mathbf{V}}(x_i)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \omega_{\mathbf{V}}\big(\varphi(t_i)\big)\big(\varphi'(t_i)\boldsymbol{\tau}_1, \varphi'(t_i)\boldsymbol{\tau}_2\big) = \big(\varphi^*\omega^2_{\mathbf{V}}\big)(t_i)(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$$

and

$$\sum_i \omega^2_{\mathbf{V}}(x_i)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \sum_i \big(\varphi^*\omega^2_{\mathbf{V}}\big)(t_i)(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) .$$

The form $\varphi^*\omega^2_{\mathbf{V}}$ is defined on a two-dimensional interval $I \subset \mathbb{R}^2$. Any 2-form in I has the form $f(t)\, \mathrm{d}t^1 \wedge \mathrm{d}t^2$, where f is a function on I depending on the form. Therefore

$$\varphi^*\omega^2_{\mathbf{V}}(t_i)(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = f(t_i)\mathrm{d}t^1 \wedge \mathrm{d}t^2(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) .$$

But $\mathrm{d}t^1 \wedge \mathrm{d}t^2(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = \tau_1^1 \cdot \tau_2^2$ is the area of the rectangle I_i spanned by the orthogonal vectors $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$.

Thus,

$$\sum_i f(t_i)\mathrm{d}t^1 \wedge \mathrm{d}t^2(\tau_1, \tau_2) = \sum_i f(t_i)|I_i| .$$

As the partition is refined we obtain in the limit

$$\int_I f(t)\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 = \int_I f(t)\,\mathrm{d}t^1\,\mathrm{d}t^2 , \qquad (13.5)$$

where, according to (13.3), the left-hand side contains the integral of the 2-form $\omega^2 = f(t)\,\mathrm{d}t^1 \wedge \mathrm{d}t^2$ over the elementary oriented surface I, and the right-hand side the integral of the function f over the rectangle I.

It remains only to recall that the coordinate representation $f(t)\,\mathrm{d}t^1 \wedge \mathrm{d}t^2$ of the form $\varphi^*\omega^2_{\mathbf{V}}$ is obtained from the coordinate expression for the form $\omega^2_{\mathbf{V}}$ by the direct substitution $x = \varphi(t)$, where $\varphi : I \to G$ is a chart of the surface S.

Carrying out this change of variable, we obtain from (13.4)

$$\mathcal{F} = \int_{S=\varphi(I)} \omega^2_{\mathbf{V}} = \int_I \varphi^*\omega^2_{\mathbf{V}} =$$

$$= \int_I \left(V^1\big(\varphi(t)\big) \begin{vmatrix} \frac{\partial x^2}{\partial t^1} & \frac{\partial x^3}{\partial t^1} \\ \frac{\partial x^2}{\partial t^2} & \frac{\partial x^3}{\partial t^2} \end{vmatrix} + V^2\big(\varphi(t)\big) \begin{vmatrix} \frac{\partial x^3}{\partial t^1} & \frac{\partial x^1}{\partial t^1} \\ \frac{\partial x^3}{\partial t^2} & \frac{\partial x^1}{\partial t^2} \end{vmatrix} + \right.$$

$$\left. + V^3\big(\varphi(t)\big) \begin{vmatrix} \frac{\partial x^1}{\partial t^1} & \frac{\partial x^2}{\partial t^1} \\ \frac{\partial x^1}{\partial t^2} & \frac{\partial x^2}{\partial t^2} \end{vmatrix} \right) \mathrm{d}t^1 \wedge \mathrm{d}t^2 .$$

This last integral, as Eq. (13.5) shows, is the ordinary Riemann integral over the rectangle I.

Thus we have found that

$$\mathcal{F} = \int_I \begin{vmatrix} V^1\big(\varphi(t)\big) & V^2\big(\varphi(t)\big) & V^3\big(\varphi(t)\big) \\ \frac{\partial \varphi^1}{\partial t^1}(t) & \frac{\partial \varphi^2}{\partial t^1}(t) & \frac{\partial \varphi^3}{\partial t^1}(t) \\ \frac{\partial \varphi^1}{\partial t^2}(t) & \frac{\partial \varphi^2}{\partial t^2}(t) & \frac{\partial \varphi^3}{\partial t^2}(t) \end{vmatrix} \mathrm{d}t^1\,\mathrm{d}t^2 , \qquad (13.6)$$

where $x = \varphi(t) = (\varphi^1, \varphi^2, \varphi^3)(t^1, t^2)$ is a chart of the surface S defining the same orientation as the field of normals we have given. If the chart $\varphi : I \to S$ gives S the opposite orientation, Eq. (13.6) does not generally hold. But, as follows from the considerations at the beginning of this subsection, the left- and right-hand sides will differ only in sign in that case.

The final formula (13.6) is obviously merely the limit of the sums of the elementary fluxes $\Delta\mathcal{F}_i \approx (\mathbf{V}(x_i), \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ familiar to us, written accurately in the coordinates t^1 and t^2.

We have considered the case of a surface defined by a single chart. In general a smooth surface can be decomposed into smooth pieces S_i having essentially no intersections with one another, and then we can find the flux through S as the sum of the fluxes though the pieces S_i.

Example 3. Suppose a medium is advancing with constant velocity $\mathbf{V} = (1, 0, 0)$. If we take any closed surface in the domain of the flow, then, since the density of the medium does not change, the amount of matter in the volume bounded by this surface must remain constant. Hence the total flux of the medium through such a surface must be zero.

In this case, let us check formula (13.6) by taking S to be the sphere $x^2 + y^2 + z^2 = R^2$.

Up to a set of area zero, which is therefore negligible, this sphere can be defined parametrically

$$\begin{aligned} x &= R\cos\psi\cos\varphi\,, \\ y &= R\cos\psi\sin\varphi\,, \\ z &= R\sin\psi\,, \end{aligned}$$

where $0 < \varphi < 2\pi$ and $-\pi/2 < \psi < \pi/2$.

After these relations and the relation $\mathbf{V} = (1, 0, 0)$ are substituted in (13.6), we obtain

$$\mathcal{F} = \int_I \begin{vmatrix} \frac{\partial x^1}{\partial t^1} & \frac{\partial x^2}{\partial t^1} \\ \frac{\partial x^1}{\partial t^2} & \frac{\partial x^2}{\partial t^2} \end{vmatrix} \mathrm{d}\varphi\,\mathrm{d}\psi = R^2 \int_{-\pi/2}^{\pi/2} \cos^2\psi\,\mathrm{d}\psi \int_0^{2\pi} \cos\varphi\,\mathrm{d}\varphi = 0\,.$$

Since the integral equals zero, we have not even bothered to consider whether it was the inward or outward flow we were computing.

Example 4. Suppose the velocity field of a medium moving in $\mathbb{R}^3$ is defined in Cartesian coordinates x, y, z by the equality $\mathbf{V}(x, y, z) = (V^1, V^2, V^3)(x, y, z) = (x, y, z)$. Let us find the flux through the sphere $x^2 + y^2 + z^2 = R^2$ into the ball that it bounds (that is, in the direction of the inward normal) in this case.

Taking the parametrization of the sphere given in the last example, and carrying out the substitution in the right-hand side of (13.6), we find that

$$\int_0^{2\pi} \mathrm{d}\varphi \int_{-\pi/2}^{\pi/2} \begin{vmatrix} R\cos\psi\cos\varphi & R\cos\psi\sin\varphi & R\sin\psi \\ -R\cos\psi\sin\varphi & R\cos\psi\cos\varphi & 0 \\ R\sin\psi\cos\varphi & -R\sin\psi\sin\varphi & R\cos\psi \end{vmatrix} \mathrm{d}\psi =$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_{-\pi/2}^{\pi/2} R^3\cos\psi\,\mathrm{d}\psi = 4\pi R^3\,.$$

We now check to see whether the orientation of the sphere given by the curvilinear coordinates (φ, ψ) agrees with that given by the inward normal. It is easy to verify that they do not agree. Hence the required flux is given by $\mathcal{F} = -4\pi R^3$.

In this case the result is easy to verify: the velocity vector $\mathbf{V}$ of the flow has magnitude equal to R at each point of the sphere, is orthogonal to the sphere, and points outward. Therefore the outward flux from the inside equals the area of the sphere $4\pi R^2$ multiplied by R. The flux in the opposite direction is then $-4\pi R^3$.

13.1.2 Definition of the Integral of a Form over an Oriented Surface

The solution of the problems considered in Subsect. 13.1.1 leads to the definition of the integral of a k-form over a k-dimensional surface.

First let S be a smooth k-dimensional surface in $\mathbb{R}^n$, defined by one standard chart $\varphi : I \to S$. Suppose a k-form ω is defined on S. The integral of the form ω over the parametrized surface $\varphi : I \to S$ is then constructed as follows.

Take a partition P of the k-dimensional standard interval $I \subset \mathbb{R}^n$ induced by partitions of its projections on the coordinate axes (closed intervals). In each interval I_i of the partition P take the vertex t_i having minimal coordinate values and attach to it the k vectors $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_k$ that go along the direction of the coordinate axes to the k vertices of I_i adjacent to t_i (Fig. 13.2). Find the vectors $\boldsymbol{\xi}_1 = \varphi'(t_i)\boldsymbol{\tau}_1, \dots, \boldsymbol{\xi}_k = \varphi'(t_i)\boldsymbol{\tau}_k$ of the tangent space $TS_{x_i=\varphi(t_i)}$, then compute $\omega(x_i)(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) =: (\varphi^*\omega)(t_i)(\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_k)$, and form the Riemann sum $\sum\limits_i \omega(x_i)(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k)$. Then pass to the limit as the mesh $\lambda(P)$ of the partition tends to zero.

Thus we adopt the following definition:

Definition 1. (Integral of a k-form ω over a given chart $\varphi : I \to S$ of a smooth k-dimensional surface.)

$$\int\limits_S \omega := \lim_{\lambda(P)\to 0} \sum_i \omega(x_i)(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \lim_{\lambda(P)\to 0} \sum_i (\varphi^*\omega)(t_i)(\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_k) . \tag{13.7}$$

If we apply this definition to the k-form $f(t)\mathrm{d}t^1 \wedge \dots \wedge \mathrm{d}t^k$ on I (when φ is the identity mapping), we obviously find that

$$\int\limits_I f(t)\mathrm{d}t^1 \wedge \dots \wedge \mathrm{d}t^k = \int\limits_I f(t)\,\mathrm{d}t^1 \cdots \mathrm{d}t^k . \tag{13.8}$$

It thus follows from (13.7) that

$$\int\limits_{S=\varphi(I)} \omega = \int\limits_I \varphi^*\omega , \tag{13.9}$$

and the last integral, as Eq. (13.8) shows, reduces to the ordinary multiple integral over the interval I of the function f corresponding to the form $\varphi^*\omega$.

We have derived the important relations (13.8) and (13.9) from Definition 1, but they themselves could have been adopted as the original definitions. In particular, if D is an arbitrary domain in $\mathbb{R}^n$ (not necessarily an interval), then, so as not to repeat the summation procedure, we set

$$\int_D f(t)\,\mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^k := \int_D f(t)\,\mathrm{d}t^1 \cdots \mathrm{d}t^k \,. \tag{13.8'}$$

and for a smooth surface given in the form $\varphi : D \to S$ and a k-form ω on it we set

$$\int_{S=\varphi(D)} \omega := \int_D \varphi^*\omega \,. \tag{13.9'}$$

If S is an arbitrary piecewise-smooth k-dimensional surface and ω is a k-form defined on the smooth pieces of S, then, representing S as the union $\bigcup\limits_i S_i$ of smooth parametrized surfaces that intersect only in sets of lower dimension, we set

$$\int_S \omega := \sum_i \int_{S_i} \omega \,. \tag{13.10}$$

In the absence of substantive physical or other problems that can be solved using (13.10), such a definition raises the question whether the magnitude of the integral of the partition $\bigcup\limits_i S_i$ is independent of the choice of the parametrization of its pieces.

Let us verify that this definition is unambiguous.

Proof. We begin by considering the simplest case in which S is a domain D_x in $\mathbb{R}^k$ and $\varphi : D_t \to D_x$ is a diffeomorphism of a domain $D_t \subset \mathbb{R}^k$ onto D_x. In $D_x = S$ the k-form ω has the form $f(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$. Then, on the one hand (13.8) implies

$$\int_{D_x} f(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k = \int_{D_x} f(x)\,\mathrm{d}x^1 \cdots dx^k \,.$$

On the other hand, by (13.9′) and (13.8′),

$$\int_{D_x} \omega := \int_{D_t} \varphi^*\omega = \int_{D_t} f\big(\varphi(t)\big) \det \varphi'(t)\,\mathrm{d}t^1 \cdots \mathrm{d}t^k \,.$$

But if $\det \varphi'(t) > 0$ in D_t, then by the theorem on change of variable in a multiple integral we have

$$\int_{D_x=\varphi(D_t)} f(x)\,\mathrm{d}x^1 \cdots \mathrm{d}x^k = \int_{D_t} f\big(\varphi(t)\big) \det \varphi'(t)\,\mathrm{d}t^1 \cdots \mathrm{d}t^k \,.$$

Hence, assuming that there were coordinates $x^1, \dots, x^k$ in $S = D_x$ and curvilinear coordinates $t^1, \dots, t^k$ of the same orientation class, we have shown that the value of the integral $\int\limits_S \omega$ is the same, no matter which of these two coordinate systems is used to compute it.

We note that if the curvilinear coordinates $t^1, \dots, t^k$ had defined the opposite orientation on S, that is, $\det \varphi'(t) < 0$, the right- and left-hand sides of the last equality would have had opposite signs. Thus, one can say that the integral is well-defined only in the case of an oriented surface of integration.

Now let $\varphi_x : D_x \to S$ and $\varphi_t : D_t \to S$ be two parametrizations of the same smooth k-dimensional surface S and ω a k-form on S. Let us compare the integrals

$$\int\limits_{D_x} \varphi_x^* \omega \quad \text{and} \quad \int\limits_{D_t} \varphi_t^* \omega \,. \tag{13.11}$$

Since $\varphi_t = \varphi_x \circ \left(\varphi_x^{-1} \circ \varphi_t\right) = \varphi_x \circ \varphi$, where $\varphi = \varphi_x^{-1} \circ \varphi_t : D_t \to D_x$ is a diffeomorphism of D_t onto D_x, it follows that $\varphi_t^* \omega = \varphi^*(\varphi_x^* \omega)$ (see Eq. (12.30) of Sect. 12.5). Hence one can obtain the form $\varphi_t^* \omega$ in D_t by the change of variable $x = \varphi(t)$ in the form $\varphi_x^* \omega$. But, as we have just verified, in this case the integrals (13.11) are equal if $\det \varphi'(t) > 0$ and differ in sign if $\det \varphi'(t) < 0$.

Thus it has been shown that if $\varphi_t : D_t \to S$ and $\varphi_x : D_x \to S$ are parametrizations of the surface S belonging to the same orientation class, the integrals (13.11) are equal. The fact that the integral is independent of the choice of curvilinear coordinates on the surface S has now been verified.

The fact that the integral (13.10) over an oriented piecewise-smooth surface S is independent of the method of partitioning $\bigcup\limits_i S_i$ into smooth pieces follows from the additivity of the ordinary multiple integral (it suffices to consider a finer partition obtained by superimposing two partitions and verify that the value of the integral over the finer partition equals the value over each of the two original partitions). □

On the basis of these considerations, it now makes sense to adopt the following chain of formal definitions corresponding to the construction of the integral of a form explained in Definition 1.

Definition 1′. (Integral of a form over an oriented surface $S \subset \mathbb{R}^n$.)

a) If the form $f(x)\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k$ is defined in a domain $D \subset \mathbb{R}^k$, then

$$\int\limits_D f(x)\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k := \int\limits_D f(x)\,\mathrm{d}x^1 \cdots \mathrm{d}x^k \,.$$

b) If $S \subset \mathbb{R}^n$ is a smooth k-dimensional oriented surface, $\varphi : D \to S$ is a parametrization of it, and ω is a k-form on S, then

$$\int_S \omega := \pm \int_D \varphi^* \omega ,$$

where the $+$ sign is taken if the parametrization φ agrees with the given orientation of S and the $-$ sign in the opposite case.

c) If S is a piecewise-smooth k-dimensional oriented surface in $\mathbb{R}^n$ and ω is a k-form on S (defined where S has a tangent plane), then

$$\int_S \omega := \sum_i \int_{S_i} \omega ,$$

where $S_1, \dots, S_m, \dots$ is a decomposition of S into smooth parametrizable k-dimensional pieces intersecting at most in piecewise-smooth surfaces of smaller dimension.

We see in particular that changing the orientation of a surface leads to a change in the sign of the integral.

13.1.3 Problems and Exercises

1. a) Let x, y be Cartesian coordinates on the plane $\mathbb{R}^2$. Exhibit the vector field whose work form is $\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$.

b) Find the integral of the form ω in a) along the following paths γ_i:

$$[0, \pi] \ni t \xrightarrow{\gamma_1} (\cos t, \sin t) \in \mathbb{R}^2 ; \quad [0, \pi] \ni t \xrightarrow{\gamma_2} (\cos t, -\sin t) \in \mathbb{R}^2 ;$$

γ_3 consists of a motion along the closed intervals joining the points $(1, 0)$, $(1, 1)$, $(-1, 1)$, $(-1, 0)$ in that order; γ_4 consists of a motion along the closed intervals joining $(1, 0)$, $(1, -1)$, $(-1, -1)$, $(-1, 0)$ in that order.

2. Let f be a smooth function in the domain $D \subset \mathbb{R}^n$ and γ a smooth path in D with initial point $p_0 \in D$ and terminal point $p_1 \in D$. Find the integral of the form $\omega = df$ over γ.

3. a) Find the integral of the form $\omega = dy \wedge dz + dz \wedge dx$ over the boundary of the standard unit cube in $\mathbb{R}^3$ oriented by an outward-pointing normal.

b) Exhibit a velocity field for which the form ω in a) is the flux form.

4. a) Let x, y, z be Cartesian coordinates in $\mathbb{R}^n$. Exhibit a velocity field for which the flux form is

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} .$$

b) Find the integral of the form ω in a) over the sphere $x^2 + y^2 + z^2 = R^2$ oriented by the outward normal.

c) Show that the flux of the field $\frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ across the sphere $(x-2)^2 + y^2 + z^2 = 1$ is zero.

d) Verify that the flux of the field in c) across the torus whose parametric equations are given in Example 4 of Sect. 12.1 is also zero.

5. It is known that the pressure P, volume V, and temperature T of a given quantity of a substance are connected by an equation $f(P,V,T) = 0$, called the *equation of state* in thermodynamics. For example, for one mole of an ideal gas the equation of state is given by Clapeyron's formula $\frac{PV}{T} - R = 0$, where R is the universal gas constant.

Since P, V, T are connected by the equation of state, knowing any pair of them, one can theoretically determine the remaining one. Hence the state of any system can be characterized, for example, by points (V,P) of the plane $\mathbb{R}^2$ with coordinates V, P. Then the evolution of the state of the system as a function of time will correspond to some path γ in this plane.

Suppose the gas is located in a cylinder in which a frictionless piston can move. By changing the position of the piston, we can change the state of the gas enclosed by the piston and the cylinder walls at the cost of doing mechanical work. Conversely, by changing the state of the gas (heating it, for example) we can force the gas to do mechanical work (lifting a weight by expanding, for example). In this problem and in Problems 6, 7, and 8 below, all processes are assumed to take place so slowly that the temperature and pressure are able to average out at each particular instant of time; thus at each instant of time the system satisfies the equation of state. These are the so-called *quasi-static processes*.

a) Let γ be a path in the VP-plane corresponding to a quasi-static transition of the gas enclosed by the piston and the cylinder walls from state V_0, P_0 to V_1, P_1. Show that the quantity A of mechanical work performed on this path is defined by the line integral $A = \int\limits_\gamma P\,\mathrm{d}V$.

b) Find the mechanical work performed by one mole of an ideal gas in passing from the state V_0, P_0 to state V_1, P_1 along each of the following paths (Fig. 13.3): γ_{OLI}, consisting of the isobar OL $(P = P_0)$ followed by the isochore LI $(V = V_1)$; γ_{OKI}, consisting of the isochore OK $(V = V_0)$ followed by the isobar KI $(P = P_1)$; γ_{OI}, consisting of the isotherm $T = \text{const}$ (assuming that $P_0V_0 = P_1V_1$).

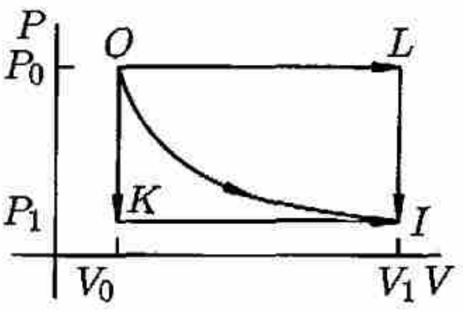


Fig. 13.3.

c) Show that the formula obtained in a) for the mechanical work performed by the gas enclosed by the piston and the cylinder walls is actually general, that is, it remains valid for the work of a gas enclosed in any deformable container.

6. The quantity of heat acquired by a system in some process of varying its states, like the mechanical work performed by the system (see Problem 5), depends not only on the initial and final states of the system, but also on the transition path. An important characteristic of a substance and the thermodynamic process performed by (or on) it is its *heat capacity*, the ratio of the heat acquired by the substance to the change in its temperature. A precise definition of heat capacity can be given as follows. Let x be a point in the plane of states F (with coordinates V, P or V, T or P, T) and $\mathbf{e} \in TF_x$ a vector indicating the direction of displacement from the point x. Let t be a small parameter. Let us consider the displacment from the state x to the state $x + t\mathbf{e}$ along the closed interval in the plane F whose endpoints are these states. Let $\Delta Q(x, t\mathbf{e})$ be the heat acquired by the substance in this process and $\Delta T(x, t\mathbf{e})$ the change in the temperature of the substance.

The *heat capacity* $C = C(x, \mathbf{e})$ of the substance (or system) corresponding to the state x and the direction $\mathbf{e}$ of displacement from that state is

$$C(x, t\mathbf{e}) = \lim_{t \to 0} \frac{\Delta Q(x, t\mathbf{e})}{\Delta T(x, t\mathbf{e})} .$$

In particular, if the system is thermally insulated, its evolution takes place without any exchange of heat with the surrounding medium. This is a so-called *adiabatic process*. The curve in the plane of states F corresponding to such a process is called an *adiabatic*. Hence, zero heat capacity of the system corresponds to displacement from a given state x along an adiabatic.

Infinite heat capacity corresponds to displacement along an isotherm $T = \text{const}$.

The heat capacities at constant volume $C_V = C(x, \mathbf{e}_V)$ and at constant pressure $C_P = C(x, \mathbf{e}_P)$, which correspond respectively to displacement along an isochore $V = \text{const}$ and an isobar $P = \text{const}$, are used particularly often. Experiment shows that in a rather wide range of states of a given mass of substance, each of the quantities C_V and C_P can be considered practically constant. The heat capacity corresponding to one mole of a given substance is customarily called the *molecular* heat capacity and is denoted (in contrast to the others) by upper case letters rather than lower case. We shall assume that we are dealing with one mole of a substance.

Between the quantity ΔQ of heat acquired by the substance in the process, the change ΔU in its internal energy, and the mechanical work ΔA it performs, the law of conservation of energy provides the connection $\Delta Q = \Delta U + \Delta A$. Thus, under a small displacement $t\mathbf{e}$ from state $x \in F$ the heat acquired can be found as the value of the form $\delta Q := \mathrm{d}U + P\,\mathrm{d}V$ at the point x on the vector $t\mathbf{e} \in TF_x$ (for the formula $P\,\mathrm{d}V$ for the work see Problem 5c)). Hence if T and V are regarded as the coordinates of the state and the displacement parameter (in a nonisothermal direction) is taken as T, then we can write

$$C = \lim_{t \to 0} \frac{\Delta Q}{\Delta T} = \frac{\partial U}{\partial T} + \frac{\partial U}{\partial V} \cdot \frac{\mathrm{d}V}{\mathrm{d}T} + P\frac{\mathrm{d}V}{\mathrm{d}T} .$$

The derivative $\frac{\mathrm{d}V}{\mathrm{d}T}$ determines the direction of displacment from the state $x \in F$ in the plane of states with coordinates T and V. In particular, if $\frac{\mathrm{d}V}{\mathrm{d}T} = 0$ then the displacement is in the direction of the isochore $V = \text{const}$, and we find that $C_V = \frac{\partial U}{\partial T}$. If $P = \text{const}$, then $\frac{\mathrm{d}V}{\mathrm{d}T} = \left(\frac{\partial V}{\partial T}\right)_{P=\text{const}}$. (In the general case $V = V(P, T)$ is the equation of state $f(P, V, T) = 0$ solved for V.) Hence

$$C_P = \left(\frac{\partial U}{\partial T}\right)_V + \left(\left(\frac{\partial U}{\partial V}\right)_T + P\right)\left(\frac{\partial V}{\partial T}\right)_P ,$$

where the subscripts P, V, and T on the right-hand side indicate the parameter of state that is fixed when the partial derivative is taken. Comparing the resulting expressions for C_V and C_P, we see that

$$C_P - C_V = \left(\left(\frac{\partial U}{\partial V}\right)_T + P\right)\left(\frac{\partial V}{\partial T}\right)_P .$$

By experiments on gases (the Joule[1]–Thomson experiments) it was established and then postulated in the model of an ideal gas that its internal energy depends only on the temperature, that is, $\left(\frac{\partial U}{\partial V}\right)_T = 0$. Thus for an ideal gas $C_P - C_V = P\left(\frac{\partial V}{\partial T}\right)_P$. Taking account of the equation $PV = RT$ for one mole of an ideal gas, we obtain the relation $C_P - C_V = R$ from this, known as *Mayer's equation*[2] in thermodynamics.

The fact that the internal energy of a mole of gas depends only on temperature makes it possible to write the form δQ as

$$\delta Q = \frac{\partial U}{\partial T}\,\mathrm{d}T + P\,\mathrm{d}V = C_V\,\mathrm{d}T + P\,\mathrm{d}V .$$

To compute the quantity of heat acquired by a mole of gas when its state varies over the path γ one must consequently find the integral of the form $C_V\,\mathrm{d}T + P\,\mathrm{d}V$ over γ. It is sometimes convenient to have this form in terms of the variables V and P. If we use the equation of state $PV = RT$ and the relation $C_P - C_V = R$, we obtain

$$\delta Q = C_P \frac{P}{R}\,\mathrm{d}V + C_V \frac{V}{R}\,\mathrm{d}P .$$

a) Write the formula for the quantity Q of heat acquired by a mole of gas as its state varies along the path γ in the plane of states F.

b) Assuming the quantities C_P and C_V are constant, find the quantity Q corresponding to the paths γ_{OLI}, γ_{OKI}, and γ_{OI} in Problem 5b).

c) Find (following Poisson) the *equation of the adiabatic* passing through the point (V_0, P_0) in the plane of states F with coordinates V and P. (Poisson found that $PV^{C_P/C_V} = \text{const}$ on an adiabatic. The quantity C_P/C_V is the *adiabatic constant* of the gas. For air $C_P/C_V \approx 1.4$.) Now compute the work one must do in order to confine a thermally isolated mole of air in the state (V_0, P_0) to the volume $V_1 = \frac{1}{2}V_0$.

7. We recall that a *Carnot cycle*[3] of variation in the state of the working body of a heat engine (for example, the gas under the piston in a cylinder) consists of the following (Fig. 13.4). There are two energy-storing bodies, a heater and a cooler (for

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[1] G.P. Joule (1818–1889) – British physicist who discovered the law of thermal action of a current and also determined, independently of Mayer, the mechanical equivalent of heat.

[2] J.P. Mayer (1814–1878) – German scholar, a physician by training; he stated the law of conservation and transformation of energy and found the mechanical equivalent of heat.

[3] N.L.S. Carnot (1796–1832) – French engineer, one of the founders of thermodynamics.

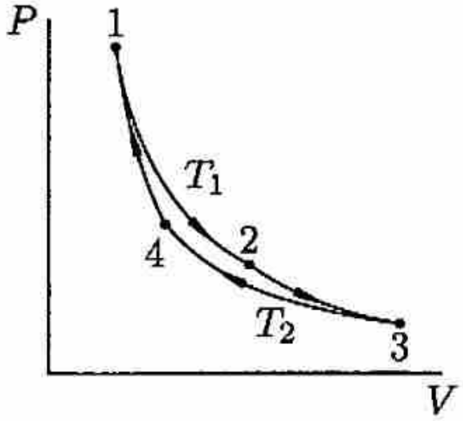

Fig. 13.4.

example, a steam boiler and the atmosphere) maintained at constant temperatures T_1 and T_2 respectively ($T_1 > T_2$). The working body (gas) of this heat engine, having temperature T_1 in State 1, is brought into contact with the heater, and by decreasing the external pressure along an isotherm, expands quasi-statically and moves to State 2. In the process the engine borrows a quantity of heat Q_1 from the heater and performs mechanical work A_{12} against the external pressure. In State 2 the gas is thermally insulated and forced to expand quasi-statically to state 3, until its temperature reaches T_2, the temperature of the cooler. In this process the engine also performs a certain quantity of work A_{23} against the external pressure. In State 3 the gas is brought into contact with the cooler and compressed isothermically to State 4 by increasing the pressure. In this process work is done on the gas (the gas itself performs negative work A_{34}), and the gas gives up a certain quantity of heat Q_2 to the cooler. State 4 is chosen so that it is possible to return from it to State 1 by a quasi-static compression along an adiabatic. Thus the gas is returned to State 1. In the process it is necessary to perform some work on the gas (and the gas itself performs negative work A_{41}). As a result of this cyclic process (a Carnot cycle) the internal energy of the gas (the working body of the engine) obviously does not change (after all, we have returned to the initial state). Therefore the work performed by the engine is $A = A_{12} + A_{23} + A_{34} + A_{41} = Q_1 - Q_2$.

The heat Q_1 acquired from the heater went only partly to perform the work A. It is natural to call the quantity $\eta = \frac{A}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$ the *efficiency* of the heat engine under consideration.

a) Using the results obtained in a) and c) of Problem 6, show that the equality $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$ holds for a Carnot cycle.

b) Now prove the following theorem, the first of Carnot's two famous theorems. *The efficiency of a heat engine working along a Carnot cycle depends only on the temperatures T_1 and T_2 of the heater and cooler.* (It is independent of the structure of the engine or the form of its working body.)

8. Let γ be a closed path in the plane of states F of the working body of an arbitrary heat engine (see Problem 7) corresponding to a closed cycle of work performed by it. The quantity of heat that the working body (a gas, for example) exchanges with the surrounding medium and the temperature at which the heat exchange takes place are connected by the *Clausius inequality* $\int\limits_{\gamma} \frac{\delta Q}{T} \leq 0$. Here δQ is the heat exchange form mentioned in Problem 6.

a) Show that for a Carnot cycle (see Problem 7), the Clausius inequality becomes equality.

b) Show that if the work cycle γ can be run in reverse, then the Clausius inequality becomes equality.

c) Let γ_1 and γ_2 be the parts of the path γ on which the working body of a heat engine acquires heat from without and imparts it to the surrounding medium respectively. Let T_1 be the maximal temperature of the working body on γ_1 and T_2 its minimal temperature on γ_2. Finally, let Q_1 be the heat acquired on γ_1 and Q_2 the heat given up on γ_2. Based on Clausius' inequality, show that $\frac{Q_2}{Q_1} \leq \frac{T_2}{T_1}$.

d) Obtain the estimate $\eta \leq \frac{T_1 - T_2}{T_1}$ for the efficiency of any heat engine (see Problem 7). This is *Carnot's second theorem*. (Estimate separately the efficiency of a steam engine in which the maximal temperature of the steam is at most 150° C, that is, $T_1 = 423\,\mathrm{K}$, and the temperature of the cooler – the surrounding medium – is of the order 20° C, that is $T_2 = 291\,\mathrm{K}$.)

e) Compare the results of Problems 7b) and 8d) and verify that a heat engine working in a Carnot cycle has the maximum possible efficiency for given values of T_1 and T_2.

9. The differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$ is said to have variables separable. It is usually rewritten in the form $g(y)\,\mathrm{d}y = f(x)\,\mathrm{d}x$, in which "the variables are separated," then "solved" by equating the primitives $\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$. Using the language of differential forms, now give a detailed mathematical explanation for this algorithm.

13.2 The Volume Element. Integrals of First and Second Kind

13.2.1 The Mass of a Lamina

Let S be a lamina in Euclidean space $\mathbb{R}^n$. Assume that we know the density $\rho(x)$ (per unit area) of the mass distribution on S. We ask how one can determine the total mass of S.

In order to solve this problem it is necessary first of all to take account of the fact that the surface density $\rho(x)$ is the limit of the ratio Δm of the quantity of mass on a portion of the surface in a neighborhood of x to the area $\Delta\sigma$ of that same portion of the surface, as the neighborhood is contracted to x.

By breaking S into small pieces S_i and assuming that ρ is continuous on S, we can find the mass of S_i, neglecting the variation of ρ within each small piece, from the relation

$$\Delta m_i \approx \rho(x_i)\,\Delta\sigma_i\ ,$$

in which $\Delta\sigma_i$ is the area of the surface S_i and $x_i \in S_i$.

Summing these approximate equalities and passing to the limit as the partition is refined, we find that

$$m = \int_S \rho \, \mathrm{d}\sigma . \tag{13.12}$$

The symbol for integration over the surface S here obviously requires some clarification so that computational formulas can be derived from it.

We note that the statement of the problem itself shows that the left-hand side of Eq. (13.12) is independent of the orientation of S, so that the integral on the right-hand side must have the same property. At first glance this appears to contrast with the concept of an integral over a surface, which was discussed in detail in Sect. 13.1. The answer to the question that thus arises is concealed in the definition of the surface element $\mathrm{d}\sigma$, to whose analysis we now turn.

13.2.2 The Area of a Surface as the Integral of a Form

Comparing Definition 1 of Sect. 13.1 for the integral of a form with the construction that led us to the definition of the area of a surface (Sect. 12.4), we see that the area of a smooth k-dimensional surface S embedded in the Euclidean space $\mathbb{R}^n$ and given parametrically by $\varphi : D \to S$, is the integral of a form Ω, which we shall provisionally call the volume element on the surface S. It follows from relation (12.10) of Sect. 12.4 that Ω (more precisely $\varphi^*\Omega$) has the form

$$\omega = \sqrt{\det(g_{ij})(t)}\, \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^k , \tag{13.13}$$

in the curvilinear coordinates $\varphi : D \to S$ (that is, when transferred to the domain D). Here $g_{ij}(t) = \left\langle \frac{\partial \varphi}{\partial t^i}, \frac{\partial \varphi}{\partial t^j} \right\rangle$, $i, j = 1, \dots, k$.

To compute the area of S over a domain $\widetilde{D}$ in a second parametrization $\widetilde{\varphi} : \widetilde{D} \to S$, one must correspondingly integrate the form

$$\widetilde{\omega} = \sqrt{\det(\tilde{g}_{ij})(\tilde{t})}\, \mathrm{d}\tilde{t}^1 \wedge \cdots \wedge \mathrm{d}\tilde{t}^k , \tag{13.14}$$

where $\tilde{g}_{ij}(\tilde{t}) = \left\langle \frac{\partial \tilde{\varphi}}{\partial \tilde{t}^i}, \frac{\partial \tilde{\varphi}}{\partial \tilde{t}^j} \right\rangle$, $i, j = 1, \dots, k$.

We denote by ψ the diffeomorphism $\varphi^{-1} \circ \widetilde{\varphi} : \widetilde{D} \to D$ that provides the change from $\tilde{t}$ coordinates to t coordinates on S. Earlier we have computed (see Remark 5 of Sect. 12.4) that

$$\sqrt{\det(\tilde{g}_{ij})(\tilde{t})} = \sqrt{\det(g_{ij})(t)} \cdot |\det \psi'(t)| . \tag{13.15}$$

At the same time, it is obvious that

$$\psi^*\omega = \sqrt{\det(g_{ij})}\big(\psi(\tilde{t})\big), \det \psi'(\tilde{t})\, \mathrm{d}\tilde{t}^1 \wedge \cdots \wedge \mathrm{d}\tilde{t}^k . \tag{13.16}$$

Comparing the equalities (13.13)–(13.16), we see that $\psi^*\omega = \widetilde{\omega}$ if $\det \psi'(\tilde{t}) > 0$ and $\psi^*\omega = -\widetilde{\omega}$ if $\det \psi'(\tilde{t}) < 0$. If the forms ω and $\widetilde{\omega}$ were obtained from the same form Ω on S through the transfers φ^* and $\widetilde{\varphi}^*$, then we must always have the equality $\psi^*(\varphi^*\Omega) = \widetilde{\varphi}^*\Omega$ or, what is the same, $\psi^*\omega = \widetilde{\omega}$.

We thus conclude that the forms on the parametrized surface S that one must integrate in order to obtain the areas of the surface are different: they differ in sign if the parametrizations define different orientations on S; these forms are equal for parametrizations that belong to the same orientation class for the surface S.

Thus the volume element Ω on S must be determined not only by the surface S embedded in $\mathbb{R}^n$, but also by the orienation of S.

This might apprear paradoxical: in our intuition, the area of a surface should not depend on its orientation!

But after all, we arrived at the definition of the area of a parametrized surface via an integral, the integral of a certain form. Hence, if the result of our computations is to be independent of the orientation of the surface, it follows that we must integrate different forms when the orientation is different.

Let us now turn these considerations into precise definitions.

13.2.3 The Volume Element

Definition 1. If $\mathbb{R}^k$ is an oriented Euclidean space with inner product $\langle\,,\rangle$, the *volume element* on $\mathbb{R}^k$ corresponding to a particular orientation and the inner product $\langle\,,\rangle$ is the skew-symmetric k-form that assumes the value 1 on an orthonormal frame of some orientation class.

The value of the k-form on the frame $\mathbf{e}_1, \dots, \mathbf{e}_k$ obviously determines this form.

We remark also that the form Ω is determined not by an individual orthonormal frame, but only by its orientation class.

Proof. In fact, if $\mathbf{e}_1, \dots, \mathbf{e}_k$ and $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_k$ are two such frames in the same orientation class, then the transition matrix O from the second basis to the first is an orthogonal matrix with $\det O = 1$. Hence

$$\Omega(\mathbf{e}_1, \dots, \mathbf{e}_k) = \det O \cdot \Omega(\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_k) = \Omega(\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_k)\ . \ \square$$

If an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_k$ is fixed in $\mathbb{R}^k$ and $\pi^1, \dots, \pi^k$ are the projections of $\mathbb{R}^k$ on the corresponding coordinate axes, obviously $\pi^1 \wedge \cdots \wedge \pi^k(\mathbf{e}_1, \dots, \mathbf{e}_k) = 1$ and

$$\Omega = \pi^1 \wedge \cdots \wedge \pi^k\ .$$

Thus,

$$\Omega(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \begin{vmatrix} \xi_1^1 & \cdots & \xi_1^k \\ \cdots & \cdots & \cdots \\ \xi_k^1 & \cdots & \xi_k^k \end{vmatrix} .$$

This is the oriented volume of the parallelepiped spanned by the ordered set of vectors $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$.

Definition 2. If the smooth k-dimensional oriented surface S is embedded in a Euclidean space $\mathbb{R}^n$, then each tangent plane TS_x to S has an orientation consistent with the orientation of S and an inner product induced by the the inner product in $\mathbb{R}^n$; hence there is a volume element $\Omega(x)$. The k-form Ω that arises on S in this way is the *volume element on* S induced by the embedding of S in $\mathbb{R}^n$.

Definition 3. The *area of an oriented smooth surface* is the integral over the surface of the volume element corresponding to the orientation chosen for the surface.

This definition of area, stated in the language of forms and made precise, is of course in agreement with Definition 1 of Sect. 12.4, which we arrived at by consideration of a smooth k-dimensional surface $S \subset \mathbb{R}^n$ defined in parametric form.

Proof. Indeed, the parametrization orients the surface and all its tangent planes TS_x. If $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$ is a frame of a fixed orientation class in TS_x, it follows from Definitions 2 and 3 for the volume element Ω that $\Omega(x)(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) > 0$. But then (see Eq. (12.7) of Sect. 12.4)

$$\Omega(x)(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) = \sqrt{\det\left(\langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_j \rangle\right)} . \ \square \tag{13.17}$$

We note that the form $\Omega(x)$ itself is defined on any set $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$ of vectors in TS_x, but Eq. (13.17) holds only on frames of a given orientation class in TS_x.

We further note that the volume element is defined only on an oriented surface, so that it makes no sense, for example, to talk about the volume element on a Möbius band in $\mathbb{R}^3$, although it does make sense to talk about the volume element of each orientable piece of this surface.

Definition 4. Let S be a k-dimensional piecewise-smooth surface (orientable or not) in $\mathbb{R}^n$, and $S_1, \dots, S_m, \dots$ a finite or countable number of smooth parametrized pieces of it intersecting at most in surfaces of dimension not larger than $k-1$ and such that $S = \bigcup\limits_i S_i$.

The *area* (or *k-dimensional volume*) of S is the sum of the areas of the surfaces S_i.

In this sense we can speak of the area of a Möbius band in $\mathbb{R}^3$ or, what is the same, try to find its mass if it is a material surface with matter having unit density.

The fact that Definition 4 is unambiguous (that the area obtained is independent of the partition $S_1, \ldots, S_m, \ldots$ of the surface) can be verified by traditional reasoning.

13.2.4 Expression of the Volume Element in Cartesian Coordinates

Let S be a smooth hypersurface (of dimension $n-1$) in an oriented Euclidean space $\mathbb{R}^n$ endowed with a continuous field of unit normal vectors $\boldsymbol{\eta}(x)$, $x \in S$, which orients it. Let V be the n-dimensional volume in $\mathbb{R}^n$ and Ω the $(n-1)$-dimensional volume element on S.

If we take a frame $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}$ in the tangent space TS_x from the orientation class determined by the unit normal $\mathbf{n}(x)$ to TS_x, we can obviously write the following equality:

$$V(x)(\boldsymbol{\eta}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}) = \Omega(x)(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}) \,. \tag{13.18}$$

Proof. This fact follows from the fact that under the given hypotheses both sides are nonnegative and equal in magnitude because the volume of the parallelepiped spanned by $\boldsymbol{\eta}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}$ is the area of the base $\Omega(x)(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1})$ multiplied by the height $|\boldsymbol{\eta}| = 1$. □

But,

$$V(x)(\boldsymbol{\eta}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}) = \begin{vmatrix} \eta^1 & \cdots & \eta^n \\ \xi_1^1 & \cdots & \xi_1^n \\ \cdots & \cdots & \cdots \\ \xi_{n-1}^1 & \cdots & \xi_{n-1}^n \end{vmatrix} =$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \eta^i(x)\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}) \,.$$

Here the variables $x^1, \ldots, x^n$ are Cartesian coordinates in the orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ that defines the orientation, and the frown over the differential dx^i indicates that it is to be omitted.

Thus we obtain the following coordinate expression for the volume element on the oriented hypersurface $S \subset \mathbb{R}^n$:

$$\Omega = \sum_{i=1}^{n} (-1)^{i-1} \eta^i(x)\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}) \,. \tag{13.19}$$

At this point it is worthwhile to remark that when the orientation of the susrface is reversed, the direction of the normal $\boldsymbol{\eta}(x)$ reverses, that is, the form Ω is replaced by the new form $-\Omega$.

It follows from the same geometric considerations that for a fixed value of $i \in \{1, \dots, n\}$

$$\langle \boldsymbol{\eta}(x), \mathrm{e}_i \rangle \Omega(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}) = V(x)(\mathrm{e}_i, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}) \,. \tag{13.20}$$

This last equality means that

$$\eta^i(x)\Omega(x) = (-1)^{i-1} \mathrm{d}x^1 \wedge \dots \wedge \widehat{\mathrm{d}x^i} \wedge \dots \wedge \mathrm{d}x^n(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}) \,. \tag{13.21}$$

For a two-dimensional surface S in $\mathbb{R}^n$ the volume element is most often denoted $\mathrm{d}\sigma$ or $\mathrm{d}S$. These symbols should not be interpreted as the differentials of some forms σ and S; they are only symbols. If x, y, z are Cartesian coordinates on $\mathbb{R}^3$, then in this notation relations (13.19) and (13.21) can be written as follows:

$$\mathrm{d}\sigma = \cos\alpha_1 \,\mathrm{d}y \wedge \mathrm{d}z + \cos\alpha_2 \mathrm{d}z \wedge \mathrm{d}x + \cos\alpha_3 \,\mathrm{d}x \wedge \mathrm{d}y \,,$$

$$\begin{aligned} \cos\alpha_1 \,\mathrm{d}\sigma &= \mathrm{d}y \wedge \mathrm{d}z \,, \\ \cos\alpha_2 \,\mathrm{d}\sigma &= \mathrm{d}z \wedge \mathrm{d}x \,, \\ \cos\alpha_3 \,\mathrm{d}\sigma &= \mathrm{d}x \wedge \mathrm{d}y \,, \end{aligned} \quad \begin{array}{l} \text{(oriented areas of the projections} \\ \text{on the coordinate planes).} \end{array}$$

Here $(\cos\alpha_1, \cos\alpha_2, \cos\alpha_3)(x)$ are the direction cosines (coordinates) of the unit normal vector $\boldsymbol{\eta}(x)$ to S at the point $x \in S$. In these equalities (as also in (13.19) and (13.21)) it would of course have been more correct to place the restriction sign $\big|_S$ on the right-hand side so as to avoid misunderstanding. But, in order not to make the formulas cumbersome, we confine ourselves to this remark.

13.2.5 Integrals of First and Second Kind

Integrals of type (13.12) arise in a number of problems, a typical representative of which is the problem consisdered above of determining the mass of a surface whose density is known. These integrals are often called integrals over a surface or integrals of first kind.

Definition 5. The *integral of a function ρ over an oriented surface S* is the integral

$$\int\limits_S \rho\Omega \tag{13.22}$$

of the differential form $\rho\Omega$, where Ω is the volume element on S (corresponding to the orientation of S chosen in the computation of the integral).

It is clear that the integral (13.22) so defined is independent of the orientation of S, since a reversal of the orientation is accompanied by a corresponding replacement of the volume element.

We emphasize that it is not really a matter of integrating a function, but rather integrating a form $\rho\Omega$ of special type over the surface S with the volume element defined on it.

Definition 6. If S is a piecewise-smooth (orientable or non-orientable) surface and ρ is a function on S, then the *integral* (13.22) *of* ρ *over the surface* S is the sum $\sum\limits_i \int\limits_{S_i} \rho\Omega$ of the integrals of ρ over the parametrized pieces $S_1, \dots, S_m, \dots$ of the partition of S described in Definition 4.

The integral (13.22) is usually called a *surface integral of first kind.*

For example, the integral (13.12), which expresses the mass of the surface S in terms of the density ρ of the mass distribution over the surface, is such an integral.

To distinguish integrals of first kind, which are independent of the orientation of the surface, we often refer to integrals of forms over an oriented surface as *surface integrals of second kind.*

We remark that, since all skew-symmetric forms on a vector space whose degrees are equal to the dimension of the space are multiples of one another, there is a connection $\omega = \rho\Omega$ between any k-form ω defined on a k-dimensional orientable surface S and the volume element Ω on S. Here ρ is some function on S depending on ω. Hence

$$\int\limits_S \omega = \int\limits_S \rho\Omega ,$$

That is, every integral of second kind can be written as a suitable integral of first kind.

Example 1. The integral (13.2′) of Sect. 13.1, which expresses the work on the path $\gamma : [a, b] \to \mathbb{R}^n$, can be written as the integral of first kind

$$\int\limits_\gamma \langle \mathbf{F}, \mathbf{e} \rangle \, \mathrm{d}s , \tag{13.23}$$

where s is arc length on γ, $\mathrm{d}s$ is the element of length (a 1-form), and $\mathbf{e}$ is a unit velocity vector containing all the information about the orientation of γ. From the point of view of the physical meaning of the problem solved by the integral (13.23), it is just as informative as the integral (13.1) of Sect. 13.1.

Example 2. The flux (13.3) of Sect. 13.1 of the velocity field $\mathbf{V}$ across a surface $S \subset \mathbb{R}^n$ oriented by unit normals $\mathbf{n}(x)$ can be written as the surface integral of first kind

$$\int\limits_{\gamma} \langle \mathbf{V}, \mathbf{n} \rangle \, \mathrm{d}\sigma \, . \tag{13.24}$$

The information about the orientation of S here is contained in the direction of the field of normals $\mathbf{n}$.

The geometric and physical content of the integrand in (13.24) is just as transparent as the corresponding meaning of the integrand in the final computational formula (13.6) of Sect. 13.1.

For the reader's information we note that quite frequently one encounters the notation $\mathrm{d}\mathbf{s} := \mathbf{e}\,\mathrm{d}s$ and $\mathrm{d}\boldsymbol{\sigma} := \mathbf{n}\,\mathrm{d}\sigma$, which introduce a vector element of length and a vector element of area. In this notation the integrals (13.23) and (13.24) have the form

$$\int\limits_{\gamma} \langle \mathbf{F}, \mathrm{d}\mathbf{s} \rangle \qquad \text{and} \qquad \int\limits_{\gamma} \langle \mathbf{V}, \mathrm{d}\boldsymbol{\sigma} \rangle \, ,$$

which are very convenient from the point of view of physical interpretation. For brevity the inner product $\langle \mathbf{A}, \mathbf{B} \rangle$ of the vectors $\mathbf{A}$ and $\mathbf{B}$ is often written $\mathbf{A} \cdot \mathbf{B}$.

Example 3. Faraday's law[4] asserts that the electromotive force arising in a closed conductor Γ in a variable magnetic field $\mathbf{B}$ is proportional to the rate of variation of the flux of the magnetic field across a surface S bounded by Γ. Let $\mathbf{E}$ be the electric field intensity. A precise statement of Faraday's law can be given as the equality

$$\oint\limits_{\Gamma} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\partial}{\partial t} \int\limits_{S} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\sigma} \, .$$

The circle in the integration sign over Γ is an additional reminder that the integral is being taken over a closed curve. The work of the field over a closed curve is often called the *circulation of the field* along this curve. Thus by Faraday's law the circulation of the electric field intensity generated in a closed conductor by a variable magnetic field equals the rate of variation of the flux of the magnetic field across a surface S bounded by Γ, taken with a suirable sign.

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[4] M. Faraday (1791–1867) – outstanding British physicist, creator of the concept of an electromagnetic field.

Example 4. Ampère's law[5]

$$\oint_{\Gamma} \mathbf{B} \cdot \mathrm{d}\mathbf{s} = \frac{1}{\varepsilon_0 c^2} \int_S \mathbf{j} \cdot \mathrm{d}\boldsymbol{\sigma}$$

(where $\mathbf{B}$ is the magnetic field intensity, $\mathbf{j}$ is the current density vector, and ε_0 and c are dimensioning constants) asserts that the circulation of the intensity of a magnetic field generated by an electric current along a contour Γ is proportional to the strength of the current flowing across the surface S bounded by the contour.

We have studied integrals of first and second kind. The reader might have noticed that this terminological distinction is very artificial. In reality we know how to integrate, and we do integrate, only differential forms. No integral is ever taken of anything else (if the integral is to claim independence of the choice of the coordinate system used to compute it).

13.2.6 Problems and Exercises

1. Give a formal proof of Eqs. (13.18) and (13.20).

2. Let γ be a smooth curve and $\mathrm{d}s$ the element of arc length on γ.

a) Show that

$$\left| \int_{\gamma} f(s)\,\mathrm{d}s \right| \le \int_{\gamma} |f(s)|\,\mathrm{d}s$$

for any function f on γ for which both integrals are defined.

b) Verify that if $|f(s)| \le M$ on γ and l is the length of γ, then

$$\left| \int_{\gamma} f(x)\,\mathrm{d}s \right| \le Ml\,.$$

c) State and prove assertions analogous to a) and b) in the general case for an integral of first kind taken over a k-dimensional smooth surface.

3. a) Show that the coordinates (x_0^1, x_0^2, x_0^3) of the center of masses distributed with linear density $\rho(x)$ along the curve γ should be given by the relations

$$x_0^i \int_{\gamma} \rho(x)\,\mathrm{d}s = \int_{\gamma} x^i \rho(x)\,\mathrm{d}s\,, \quad i = 1,2,3\,.$$

b) Write the equation of a helix in $\mathbb{R}^3$ and find the coordinates of the center of mass of a piece of this curve, assuming that the mass is distributed along the curve with constant density equal to 1.

[5] A.M. Ampère (1775–1836) – French physicist and mathematician, one of the founders of modern electrodynamics.

c) Exhibit formulas for the center of masses distributed over a surface S with surface density ρ and find the center of masses that are uniformly distributed over the surface of a hemisphere.

d) Exhibit the formulas for the moment of inertia of a mass distributed with density ρ over the surface S.

e) The tire on a wheel has mass 30 kg and the shape of a torus of outer diameter 1 m and inner diameter 0.5 m. When the wheel is being balanced, it is placed on a balancing lathe and rotated to a velocity corresponding to a speed of the order of 100 km/hr, then stopped by brake pads rubbing against a steel disk of diameter 40 cm and width 2 cm. Estimate the temperature to which the disk would be heated if all the kinetic energy of the the spinning tire went into heating the disk when the wheel was stopped. Assume that the heat capacity of steel is $c = 420$ J/(kg-K).

4. a) Show that the gravitational force acting on a point mass m_0 located at (x_0, y_0, z_0) due to a material curve γ having linear density ρ is given by the formula

$$F = Gm_0 \int_\gamma \frac{\rho}{|\mathbf{r}|^3}\mathbf{r}\, ds ,$$

where G is the gravitational constant and $\mathbf{r}$ is the vector with coordinates $(x - x_0, y - y_0, z - z_0)$.

b) Write the corresponding formula in the case when the mass is distributed over a surface S.

c) Find the gravitational field of a homogeneous material line.

d) Find the gravitational field of a homogeneous material sphere. (Exhibit the field both outside the ball bounded by the sphere and inside the ball.)

e) Find the gravitational field created in space by a homogeneous material ball (consider both exterior and interior points of the ball).

f) Regarding the Earth as a liquid ball, find the pressure in it as a function of the distance from the center. (The radius of the Earth is 6400 km, and its average density is $6\,\mathrm{g/cm}^3$.)

5. Let γ_1 and γ_2 be two closed conductors along which currents J_1 and J_2 respectively are flowing. Let $d\mathbf{s}_1$ and $d\mathbf{s}_2$ be the vector elements of these conductors corresponding to the directions of current in them. Let the vector $\mathbf{R}_{12}$ be directed from $d\mathbf{s}_1$ to $d\mathbf{s}_2$, and $\mathbf{R}_{21} = -\mathbf{R}_{12}$.

According to the *Biot–Savart law*[6] the force $d\mathbf{F}_{12}$ with which the first element acts on the second is

$$d\mathbf{F}_{12} = \frac{J_1 J_2}{c_0^2|\mathbf{R}_{12}|^2}\Big[d\mathbf{s}_2, [d\mathbf{s}_1, \mathbf{R}_{12}]\Big] ,$$

where the brackets denote the vector product of the vectors and c_0 is a dimensioning constant.

a) Show that, on the level of an abstract differential form, it could happen that $d\mathbf{F}_{12} \neq -d\mathbf{F}_{21}$ in the differential Biot–Savart formula, that is, "the reaction is not equal and opposite to the action."

[6] Biot (1774–1862), Savart (1791–1841) – French physicists.

b) Write the (integral) formulas for the total forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ for the interaction of the conductors γ_1 and γ_2 and show that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

6. *The co-area formula (the Kronrod–Federer formula).*

Let M^m and N^n be smooth surfaces of dimensions m and n respectively, embedded in a Euclidean space of high dimension (M^m and N^n may also be abstract Riemannian manifolds, but that is not important at the moment). Suppose that $m \geq n$.

Let $f : M^m \to N^n$ be a smooth mapping. When $m > n$, the mapping $\mathrm{d}f(x) : T_x M^m \to T_{f(x)} N^n$ has a nonempty kernel $\ker \mathrm{d}f(x)$. Let us denote by $T_x^{\perp} M^m$ the orthogonal complement of $\ker \mathrm{d}f(x)$, and by $J(f,x)$ the Jacobian of the mapping $\mathrm{d}f(x)\Big|_{T_x^{\perp} M^m} : T_x^{\perp} M^m \to T_{f(x)} N^n$. If $m = n$, then $J(f,x)$ is the usual Jacobian.

Let $\mathrm{d}v_k(p)$ denote the volume element on a k-dimensional surface at the point p. We shall assume that $v_0(E) = \operatorname{card} E$, where $v_k(E)$ is the k-volume of E.

a) Using Fubini's theorem and the rank theorem (on the local canonical form of a smooth mapping) if necessary, prove the following formula of Kronrod and Federer: $\int\limits_{M^m} J(f,x)\,\mathrm{d}v_m(x) = \int\limits_{N^n} v_{m-n}\Big(f^{-1}(y)\Big)\,\mathrm{d}v_n(y)$.

b) Show that if A is a measurable subset of M^m, then

$$\int\limits_A J(f,x)\,\mathrm{d}v_m(x) = \int\limits_{N^n} v_{m-n}\Big(A \cap f^{-1}(y)\Big)\,\mathrm{d}v_n(y)\ .$$

This is the general Kronrod–Federer formula.

c) Prove the following strengthening of Sard's theorem (which in its simplest version asserts that the image of the set of critical points of a smooth mapping has measure zero). (See Problem 8 of Sect. 11.5.)

Suppose as before that $f : M^m \to N^n$ is a smooth mapping and K is a compact set in M^m on which $\operatorname{rank} \mathrm{d}f(x) < n$ for all $x \in K$.

Then $\int\limits_{N^n} v_{m-n}\Big(K \cap f^{-1}(y)\Big)\,\mathrm{d}v_n(y) = 0$. Use this result to obtain in addition the simplest version of Sard's theorem stated above.

d) Verify that if $f : D \to \mathbb{R}$ and $u : D \to \mathbb{R}$ are smooth functions in a regular domain $D \subset \mathbb{R}^n$ and u has no critical points in D, then

$$\int\limits_D f\,\mathrm{d}v = \int\limits_{\mathbb{R}} \mathrm{d}t \int\limits_{u^{-1}(t)} f \frac{\mathrm{d}\sigma}{|\nabla u|}\ .$$

e) Let $V_f(t)$ be the measure (volume) of the set $\{x \in D |\, f(x) > t\}$, and let the function f be nonnegative and bounded in the domain D.

Show that $\int\limits_D f\,\mathrm{d}v = -\int\limits_{\mathbb{R}} t\,\mathrm{d}V_f(t) = \int\limits_0^{\infty} V_f(t)\,\mathrm{d}t$.

f) Let $\varphi \in C^{(1)}(\mathbb{R}, \mathbb{R}_+)$ and $\varphi(0) = 0$, while $f \in C^{(1)}(D, \mathbb{R})$ and $V_{|f|}(t)$ is the measure of the set $\{x \in D \Big| \, |f(x)| > t\}$. Verify that $\int\limits_D \varphi \circ f\,\mathrm{d}v = \int\limits_0^{\infty} \varphi'(t) V_{|f|}(t)\,\mathrm{d}t$.

13.3 The Fundamental Integral Formulas of Analysis

The most important formula of analysis is the Newton–Leibniz formula (fundamental theorem of calculus). In the present section we shall obtain the formulas of Green, Gauss–Ostrogradskii, and Stokes, which on the one hand are an extension of the Newton–Leibniz formula, and on the other hand, taken together, constitute the most-used part of the machinery of integral calculus.

In the first three subsections of this section, without striving for generality in our statements, we shall obtain the three classical integral formulas of analysis using visualizable material. They will be reduced to one general Stokes formula in the fourth subsection, which can be read formally independently of the others.

13.3.1 Green's Theorem

Green's[7] theorem is the following.

Proposition 1. *Let $\mathbb{R}^2$ be the plane with a fixed coordinate grid x, y, and let $\overline{D}$ be a compact domain in this plane bounded by piecewise-smooth curves. Let P and Q be smooth functions in the closed domain $\overline{D}$. Then the following relation holds:*

$$\iint\limits_{\overline{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial \overline{D}} P\,\mathrm{d}x + Q\ \ Dy\ , \tag{13.25}$$

in which the right-hand side contains the integral over the boundary $\partial\overline{D}$ of the domain $\overline{D}$ oriented consistently with the orientation of the domain $\overline{D}$ itself.

We shall first consider the simplest version of (13.25) in which $\overline{D}$ is the square $I = \{(x,y) \in \mathbb{R}^2 \big| 0 \le x \le 1,\ 0 \le y \le 1\}$ and $Q \equiv 0$ in I. Then Green's theorem reduces to the equality

$$\iint\limits_{I} \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = -\int\limits_{\partial I} P\,\mathrm{d}x\ , \tag{13.26}$$

which we shall prove.

Proof. Reducing the double integral to an iterated integral and applying the fundamental theorem of calculus, we obtain

[7] G. Green (1793–1841) – British mathematician and mathematical physicist. Newton's grave in Westminster Abbey is framed by five smaller gravestones with brilliant names: Faraday, Thomson (Lord Kelvin), Green, Maxwell, and Dirac.

$$\iint\limits_{\overline{D}} \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \int\limits_0^1 \mathrm{d}x \int\limits_0^1 \frac{\partial P}{\partial y}\,\mathrm{d}y =$$

$$= \int\limits_0^1 \bigl(P(x,1) - P(x,2)\bigr)\,\mathrm{d}x = -\int\limits_0^1 P(x,0)\,\mathrm{d}x + \int\limits_0^1 P(x,1)\,\mathrm{d}x\ .$$

The proof is now finished. What remains is a matter of definitions and interpretation of the relation just obtained. The point is that the difference of the last two integrals is precisely the right-hand side of relation (13.26).

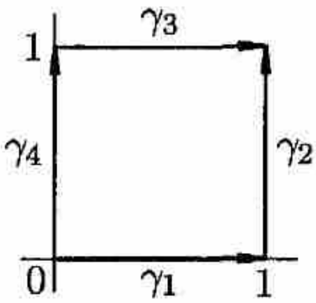

Fig. 13.5.

Indeed, the piecewise-smooth curve ∂I breaks into four pieces (Fig. 13.5), which can be regarded as parametrized curves

$$\gamma_1 : [0,1] \to \mathbb{R}^2, \text{ where } x \overset{\gamma_1}{\longmapsto} (x,0),$$
$$\gamma_2 : [0,1] \to \mathbb{R}^2, \text{ where } y \overset{\gamma_2}{\longmapsto} (1,y),$$
$$\gamma_3 : [0,1] \to \mathbb{R}^2, \text{ where } x \overset{\gamma_3}{\longmapsto} (x,1),$$
$$\gamma_4 : [0,1] \to \mathbb{R}^2, \text{ where } y \overset{\gamma_4}{\longmapsto} (0,y),$$

By definition of the integral of the 1-form $\omega = P\,\mathrm{d}x$ over a curve

$$\int\limits_{\gamma_1} P(x,y)\,\mathrm{d}x := \int\limits_{[0,1]} \gamma_1^*\bigl(P(x,y)\,\mathrm{d}x\bigr) := \int\limits_0^1 P(x,0)\,\mathrm{d}x\ ,$$

$$\int\limits_{\gamma_2} P(x,y)\,\mathrm{d}x := \int\limits_{[0,1]} \gamma_2^*\bigl(P(x,y)\,\mathrm{d}x\bigr) := \int\limits_0^1 0\,\mathrm{d}y = 0\ ,$$

$$\int\limits_{\gamma_3} P(x,y)\,\mathrm{d}x := \int\limits_{[0,1]} \gamma_3^*\bigl(P(x,y)\,\mathrm{d}x\bigr) := \int\limits_0^1 P(x,1)\,\mathrm{d}x\ ,$$

$$\int\limits_{\gamma_4} P(x,y)\,\mathrm{d}x := \int\limits_{[0,1]} \gamma_4^*\bigl(P(x,y)\,\mathrm{d}x\bigr) := \int\limits_0^1 0\,\mathrm{d}y = 0\ ,$$

and, in addition, by the choice of the orientation of the boundary of the domain, taking account of the orientations of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, it is obvious that

$$\int_{\partial I} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{-\gamma_3} \omega + \int_{-\gamma_4} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega - \int_{\gamma_3} \omega - \int_{\gamma_4} \omega ,$$

where $-\gamma_i$ is the curve γ_i taken with the orientation opposite to the one defined by γ_i.

Thus Eq. (13.26) is now verified. □

It can be verified similarly that

$$\iint_I \frac{\partial Q}{\partial x} \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial I} Q\,\mathrm{d}y , \tag{13.27}$$

Adding (13.26) and (13.27), we obtain Green's formula

$$\iint_I \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_{\partial I} P\,\mathrm{d}x + Q\,\mathrm{d}y \tag{13.25'}$$

for the square I.

We remark that the asymmetry of P and Q in Green's formula (13.25) and in Eqs. (13.26) and (13.27) comes from the asymmetry of x and y: after all, x and y are ordered, and it is that ordering that gives the orientation in $\mathbb{R}^2$ and in I.

In the language of forms, the relation (13.25′) just proved can be rewritten as

$$\int_I \mathrm{d}\omega = \int_{\partial I} \omega , \tag{13.25''}$$

where ω is an arbitrary smooth form on I. The integrand on the right-hand side here is the restriction of the form ω to the boundary ∂I of the the square I.

The proof of relation (13.26) just given admits an obvious generalization: If D_y is not a square, but a "curvilinear quadrilateral" whose lateral sides are vertical closed intervals (possibly degenerating to a point) and whose other two sides are the graphs of piecewise-smooth functions $\varphi_1(x) \leq \varphi_2(x)$ over the closed interval $[a, b]$ of the x-axis, then

$$\iint_{D_y} \frac{\partial P}{\partial y} \,\mathrm{d}x\,\mathrm{d}y = -\int_{\partial D_y} P\,\mathrm{d}x . \tag{13.26'}$$

Similarly, if there is such a "quadrilateral" D_x with respect to the y-axis, that is, having two horizontal sides, then for it we have the equality

$$\iint_{D_x} \frac{\partial Q}{\partial x} \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D_x} Q\,\mathrm{d}y . \tag{13.27'}$$

Now let us assume that the domain $\overline{D}$ can be cut into a finite number of domains of type D_y (Fig. 13.6). Then a formula of the form (13.26′) also holds for that region $\overline{D}$.

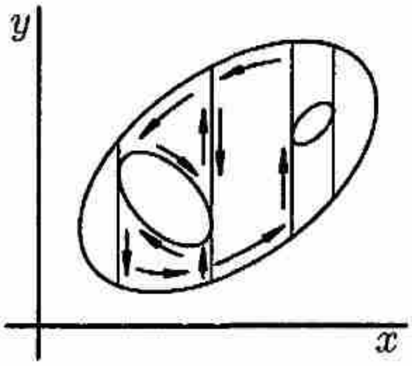

Fig. 13.6.

Proof. In fact, by additivity, the double integral over the domain $\overline{D}$ is the sum of the integrals over the pieces of type D_y into which $\overline{D}$ is divided. Formula (13.26′) holds for each such piece, that is, the double integral over that piece equals the integral of $P\,\mathrm{d}x$ over the oriented boundary of the piece. But adjacent pieces induce opposite orientations on their common boundary, so that when the integrals over the boundaries are added, all that remains after cancellation is the integral over the boundary $\partial\overline{D}$ of the domain $\overline{D}$ itself. □

Similarly, if $\overline{D}$ admits a partition into domains of type D_x, an equality of type (13.27′) holds for it.

We agree to call domains that can be cut both into pieces of type D_x and into pieces of type D_y *elementary domains*. In fact, this class is sufficiently rich for all practical applications.

By writing both relations (13.26′) and (13.27′) for a simple domain, we obtain (13.25) by adding them.

Thus, Green's theorem is proved for simple domains.

We shall not undertake any further sharpenings of Green's formula at this point (on this account see Problem 2 below), but rather demonstrate a second, very fruitful line of reasoning that one may pursue after establishing Eqs. (13.25′) and (13.25″).

Suppose the domain C has been obtained by a smooth mapping $\varphi : I \to C$ of the square I. If ω is a smooth 1-form on C, then

$$\int\limits_C \mathrm{d}\omega := \int\limits_I \varphi^* \,\mathrm{d}\omega = \int\limits_I \mathrm{d}\varphi^*\omega \overset{!}{=} \int\limits_{\partial I} \varphi^*\omega =: \int\limits_{\partial C} \omega\,. \tag{13.28}$$

The exclamation point here distinguishes the equality we have already proved (see (13.25″)); the extreme terms in these equalities are definitions or direct consequences of them; the remaining equality, the second from the left, results from the fact that exterior differentiation is independent of the coordinate system.

Hence Green's formula also holds for the domain C.

Finally, if it is possible to cut any oriented domain $\overline{D}$ into a finite number of domains of the same type as C, the considerations already described involving the mutual cancellation of the integrals over the portions of the boundaries of the C_i inside $\overline{D}$ imply that

$$\int\limits_{\overline{D}} \mathrm{d}\omega = \sum_i \int\limits_{C_i} \mathrm{d}\omega = \sum_i \int\limits_{\partial C_i} \omega = \int\limits_{\partial \overline{D}} \omega \,, \tag{13.29}$$

that is, Green's formula also holds for $\overline{D}$.

It can be shown that every domain with a piecewise-smooth boundary belongs to this last class of domains, but we shall not do so, since we shall describe below (Chap. 15) a useful technical device that makes it possible to avoid such geometric complications, replacing them by an analytic problem that is comparatively easy to solve.

Let us consider some examples of the use of Green's formula.

Example 1. Let us set $P = -y$, $Q = x$ in (13.25). We then obtain

$$\int\limits_{\partial D} -y\,\mathrm{d}x + x\,\mathrm{d}y = \int\limits_{D} 2\,\mathrm{d}x\,\mathrm{d}y = 2\sigma(D)\;,$$

where $\sigma(D)$ is the area of D. Using Green's formula one can thus obtain the following expression for the area of a domain on the plane in terms of line integrals over the oriented boundary of the domain:

$$\sigma(D) = \frac{1}{2} \int\limits_{\partial D} -y\,\mathrm{d}x + x\,\mathrm{d}y = -\int\limits_{\partial D} y\,\mathrm{d}x = \int\limits_{\partial D} x\,\mathrm{d}y\;.$$

It follows in particular from this that the work $A = \int\limits_{\gamma} P\,\mathrm{d}V$ performed by a heat engine in changing the state of its working substance over a closed cycle γ equals the area of the domain bounded by the curve γ in the PV-plane of states (see Problem 5 of Sect. 13.1).

Example 2. Let $\overline{B} = \{(x, y) \in \mathbb{R}^2 \big| \, x^2 + y^2 \leq 1\}$ be the closed disk in the plane. We shall show that any smooth mapping $f : \overline{B} \to \overline{B}$ of the closed disk into itself has at least one fixed point (that is, a point $p \in \overline{B}$ such that $f(p) = p$).

Proof. Assume that the mapping f has no fixed points. Then for every point $p \in \overline{B}$ the ray with initial point $f(p)$ passing through the point p and the point $\varphi(p) \in \partial B$ where this ray intersects the circle bounding $\overline{B}$ are uniquely determined. Thus a mapping $\varphi : \overline{B} \to \partial\overline{B}$ would arise, and it is obvious that the restriction of this mapping to the boundary would be the identity mapping. Moreover, it would have the same smoothness as the mapping f itself. We shall show that no such mapping φ can exist.

In the domain $\mathbb{R}^2 \setminus 0$ (the plane with the origin omitted) let us consider the form $\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$ that we encountered in Sect. 13.1. It can be verified immediately that $d\omega = 0$. Since $\partial\overline{B} \subset \mathbb{R}^2 \setminus 0$, given the mapping $\varphi : \overline{B} \to \partial\overline{B}$, one could obtain a form $\varphi^*\omega$ on $\overline{B}$, and $d\varphi^*\omega = \varphi^*(d\omega) = \varphi^*0 = 0$. Hence by Green's formula

$$\int\limits_{\partial\overline{B}} \varphi^*\omega = \int\limits_{\overline{B}} d\varphi^*\omega = 0 \,.$$

But the restriction of φ to $\partial\overline{B}$ is the identity mapping, and so

$$\int\limits_{\partial\overline{B}} \varphi^*\omega = \int\limits_{\partial\overline{B}} \omega \,.$$

This last integral, as was verified in Example 1 of Sect. 13.1, is nonzero. This contradiction completes the proof of the assertion. □

This assertion is of course valid for a ball of any dimension (see Example 5 below). It also holds not only for smooth mappings, but for all continuous mappings $f : \mathbb{B} \to \mathbb{B}$. In this general form it is called the *Brouwer fixed-point theorem.*[8]

13.3.2 The Gauss–Ostrogradskii Formula

Just as Green's formula connects the integral over the boundary of a plane domain with a corresponding integral over the domain itself, the Gauss–Ostrogradskii formula given below connects the integral over the boundary of a three-dimensional domain with an integral over the domain itself.

Proposition 2. *Let $\mathbb{R}^3$ be three-dimensional space with a fixed coordinate system x, y, z and $\overline{D}$ a compact domain in $\mathbb{R}^3$ bounded by piecewise-smooth surfaces. Let P, Q, and R be smooth functions in the closed domain $\overline{D}$.*

Then the following relation holds:

$$\iiint\limits_{\overline{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\ Dz = \iint\limits_{\partial\overline{D}} P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy \,. \tag{13.30}$$

[8] L.E.J. Brouwer (1881–1966) – Well-known Dutch mathematician. A number of fundamental theorems of topology are associated with his name, as well as an analysis of the foundations of mathematics that leads to the philosophico-mathematical concepts called intuitionism.

The Gauss–Ostrogradskii formula (13.30) can be derived by repeating the derivation of Green's formula step by step with obvious modifications. So as not to do a verbatim repetition, let us begin by considering not a cube in $\mathbb{R}^3$, but the domain D_x shown in Fig. 13.7, which is bounded by a lateral cylindrical surface S with generator parallel to the z-axis and two caps S_1 and S_2 which are the graphs of piecewise-smooth functions φ_1 and φ_2 defined in the same domain $G \subset \mathbb{R}^2_{xy}$. We shall verify that the relation

$$\iiint\limits_{D_z} \frac{\partial R}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{\partial D_z} R\,\mathrm{d}x \wedge \mathrm{d}y \qquad (13.31)$$

holds for D_z.

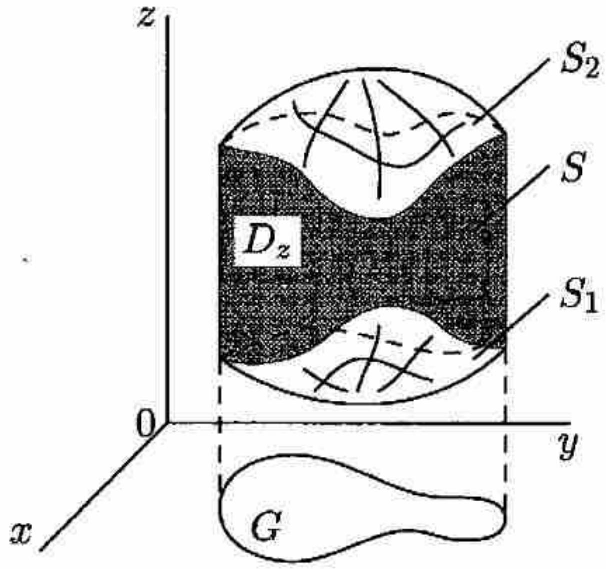

Fig. 13.7.

Proof.

$$\iiint\limits_{D_z} \frac{\partial R}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{G} \mathrm{d}x\,\mathrm{d}y \int\limits_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial R}{\partial z}\,\mathrm{d}z =$$

$$= \iint\limits_{G} \big(R(x,y,\varphi_2(x,y)) - R(x,y,\varphi_1(x,y))\big)\,\mathrm{d}x\,\mathrm{d}y =$$

$$= -\iint\limits_{G} \big(R(x,y,\varphi_1(x,y))\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{G} \big(R(x,y,\varphi_2(x,y))\,\mathrm{d}x\,\mathrm{d}y\,.$$

The surfaces S_1 and S_2 have the following parametrizations:

$$S_1 : (x,y) \longmapsto \big(x,y,\varphi_1(x,y)\big)\,,$$
$$S_2 : (x,y) \longmapsto \big(x,y,\varphi_2(x,y)\big)\,.$$

The curvilinear coordinates (x, y) define the same orientation on S_2 that is induced by the orientation of the domain D_z, and the opposite orientation on S_1. Hence if S_1 and S_2 are regarded as pieces of the boundary of D_z oriented as indicated in Prosition 2, these last two integrals can be interpreted as integrals of the form $R\,\mathrm{d}x \wedge \mathrm{d}y$ over S_1 and S_2.

The cylindrical surface S has a parametric representation $(t, z) \mapsto (x(t), y(t), z)$, so that the restriction of the form $R\,\mathrm{d}x \wedge \mathrm{d}y$ to S equals zero, and so consequently, its integral over S is also zero.

Thus relation (13.31) does indeed hold for the domain D_z. $\square$

If the oriented domain $\overline{D}$ can be cut into a finite number of domains of the type D_z, then, since adjacent pieces induce opposite orientations on their common boundary, the integrals over these pieces will cancel out, leaving only the integral over the boundary $\partial\overline{D}$.

Consequently, formula (13.31) also holds for domains that admit this kind of partition into domains of type D_z.

Similarly, one can introduce domains D_y and D_x whose cylindrical surfaces have generators parallel to the y-axis or x-axis respectively and show that if a domain $\overline{D}$ can be divided into domains of type D_y or D_x, then the relations

$$\iiint\limits_{\overline{D}} \frac{\partial Q}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{\partial\overline{D}} Q\,\mathrm{d}z \wedge \mathrm{d}x\,, \tag{13.32}$$

$$\iiint\limits_{\overline{D}} \frac{\partial P}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{\partial\overline{D}} P\,\mathrm{d}y \wedge \mathrm{d}z\,. \tag{13.33}$$

Thus, if $\overline{D}$ is a *simple domain*, that is, a domain that admits each of the three types of partitions just described into domains of types D_x, D_y, and D_z, then, by adding (13.31), (13.32), and (13.33), we obtain (13.30) for $\overline{D}$.

For the reasons given in the derivation of Green's theorem, we shall not undertake the description of the conditions for a domain to be simple or any further sharpening of what has been proved (in this connection see Problem 8 below or Example 12 in Sect. 17.5).

We note, however, that in the language of forms, the Gauss–Ostrogradskii formula can be written in coordinate-free form as follows:

$$\int\limits_{\overline{D}} \mathrm{d}\omega = \int\limits_{\partial\overline{D}} \omega\,, \tag{13.30'}$$

where ω is a smooth 2-form in $\overline{D}$.

Since formula (13.30′) holds for the cube $I = I^3 = \{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \le 1 \le 1, 0 \le y \le 1, 0 \le z \le 1\}$, as we have shown, its extension to more general classes of domains can of course be carried out using the standard computations (13.28) and (13.29).

Example 3. The law of Archimedes. Let us compute the buoyant force of a homogeneous liquid on a body D immersed in it. We choose the Cartesian coordinates x, y, z in $\mathbb{R}^3$ so that the xy-plane is the surface of the liquid and the z-axis is directed out of the liquid. A force $\rho g z \mathbf{n}\, \mathrm{d}\sigma$ is acting on an element $\mathrm{d}\sigma$ of the surface S of D located at depth z, where ρ is the density of the liquid, g is the acceleration of gravity, and $\mathbf{n}$ is a unit outward normal to the surface at the point of the surface corresponding to $\mathrm{d}\sigma$. Hence the resultant force can be expressed by the integral

$$\mathbf{F} = \iint_S \rho g z \mathbf{n}\, \mathrm{d}\sigma .$$

If $\mathbf{n} = \mathbf{e}_x \cos\alpha_x + \mathbf{e}_y \cos\alpha_y + \mathbf{e}_z \cos\alpha_z$, then $\mathbf{n}\, \mathrm{d}\sigma = \mathbf{e}_x\, \mathrm{d}y \wedge \mathrm{d}z + \mathbf{e}_y\, \mathrm{d}z \wedge \mathrm{d}x + \mathbf{e}_z\, \mathrm{d}x \wedge \mathrm{d}y$ (see Subsect. 13.2.4). Using the Gauss–Ostrogradskii formula (13.30), we thus find that

$$\begin{aligned}\mathbf{F} = \mathbf{e}_x \rho g \iint_S z\, \mathrm{d}y \wedge \mathrm{d}z + \mathbf{e}_y \rho g \iint_S z\, \mathrm{d}z \wedge \mathrm{d}x + \mathbf{e}_z \rho g \iint_S z\, \mathrm{d}x \wedge \mathrm{d}y = \\ = \mathbf{e}_x \rho g \iiint_D 0\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \mathbf{e}_y \rho g \iiint_D 0\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \\ + \mathbf{e}_z \rho g \iiint_D \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \rho g V \mathbf{e}_z ,\end{aligned}$$

where V is the volume of the body D. Hence $P = \rho g V$ is the weight of a volume of the liquid equal to the volume occupied by the body. We have arrived at Archimedes' law: $\mathbf{F} = P\mathbf{e}_z$.

Example 4. Using the Gauss–Ostrogradskii formula (13.30), one can give the following formulas for the volume $V(D)$ of a body D bounded by a surface ∂D.

$$\begin{aligned}V(D) = \frac{1}{3} \iint_{\partial D} x\, \mathrm{d}y \wedge \mathrm{d}z + y\, \mathrm{d}z \wedge \mathrm{d}x + z\, \mathrm{d}x \wedge \mathrm{d}y = \\ = \iint_{\partial D} x\, \mathrm{d}y \wedge \mathrm{d}z = \iint_{\partial D} y\, \mathrm{d}z \wedge \mathrm{d}x = \iint_{\partial D} z\, \mathrm{d}x \wedge \mathrm{d}y .\end{aligned}$$

13.3.3 Stokes' Formula in $\mathbb{R}^3$

Proposition 3. *Let S be an oriented piecewise-smooth compact two-dimensional surface with boundary ∂S embedded in a domain $G \subset \mathbb{R}^3$, in which a smooth 1-form $\omega = P\, \mathrm{d}x + Q\, \mathrm{d}y + R\, \mathrm{d}z$ is defined. Then the following relation holds:*

$$\boxed{\begin{aligned}\int_{\partial S} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z \,+ \\ + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y\,,\end{aligned}} \tag{13.34}$$

where the orientation of the boundary ∂S is chosen consistently with the orientation of the surface S.

In other notation, this means that

$$\int_S \mathrm{d}\omega = \int_{\partial \omega} \omega\,. \tag{13.34'}$$

Proof. If C is a standard parametrized surface $\varphi : I \to C$ in $\mathbb{R}^3$, where I is a square in $\mathbb{R}^2$, relation (13.34) follos from Eqs. (13.28) taking account of what has been proved for the square and Green's formula.

If the orientable surface S can be cut into elementary surfaces of this type, then relation (13.34) is also valid for it, as follows from Eqs. (13.29) with $\overline{D}$ replaced by S. □

As in the preceding cases, we shall not prove at this point that, for example, a piecewise-smooth surface admits such a partition.

Let us show what this proof of formula (13.34) would look like in coordinate notation. To avoid expressions that are really too cumbersome, we shall write out only the first, main part of its two expressions, and with some simplifications even in that. To be specific, let us introduce the notation x^1, x^2, x^3 for the coordinates of a point $x \in \mathbb{R}^3$ and verify only that

$$\int_{\partial S} P(x)\,\mathrm{d}x^1 = \iint_S \frac{\partial P}{\partial x^2}\,\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial P}{\partial x^3}\,\mathrm{d}x^3 \wedge \mathrm{d}x^1\,,$$

since the other two terms on the left-hand side of (13.34) can be studied s imilarly. For simplicity we shall assume that S can be obtained by a smooth

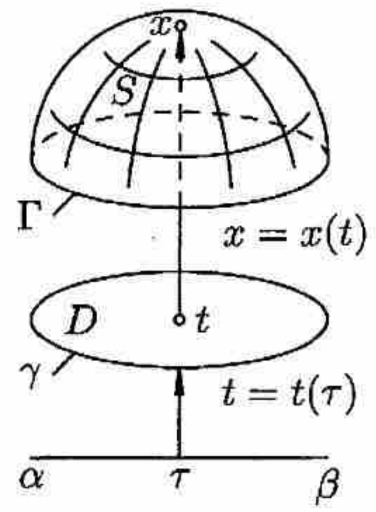

Fig. 13.8.

mapping $x = x(t)$ of a domain D in the plane $\mathbb{R}^2$ of the variables t^1, t^2 and bounded by a smooth curve $\gamma = \partial D$ parametrized via a mapping $t = t(\tau)$ by the points of the closed interval $\alpha \le \tau \le \beta$ (Fig. 13.8). Then the boundary $\Gamma = \partial S$ of the surface S can be written as $x = x\big(t(\tau)\big)$, where τ ranges over the closed interval $[\alpha, \beta]$. Using the definition of the integral over a curve, Green's formula for a plane domain D, and the definition of the integral over a parametrized surface, we find successively

$$\int_\Gamma P(x)\,\mathrm{d}x^1 := \int_\alpha^\beta P\big(x(t(\tau))\big)\Big(\frac{\partial x^1}{\partial t^1}\frac{\mathrm{d}t^1}{\mathrm{d}\tau} + \frac{\partial x^1}{\partial t^2}\frac{\mathrm{d}t^2}{\mathrm{d}\tau}\Big)\,\mathrm{d}\tau =$$

$$= \int_\gamma \Big(P\big(x(t)\big)\frac{\partial x^1}{\partial t^1}\Big)\mathrm{d}t^1 + \Big(P\big(x(t)\big)\frac{\partial x^1}{\partial t^2}\Big)\mathrm{d}t^2 \overset{!}{=}$$

$$\overset{!}{=} \iint_D \Big[\frac{\partial}{\partial t^1}\Big(P\frac{\partial x^1}{\partial t^2}\Big) - \frac{\partial}{\partial t^2}\Big(P\frac{\partial x^1}{\partial t^1}\Big)\Big]\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 =$$

$$= \iint_D \Big(\frac{\partial P}{\partial t^1}\frac{\partial x^1}{\partial t^2} - \frac{\partial P}{\partial t^2}\frac{\partial x^1}{\partial t^1}\Big)\mathrm{d}t^1 \wedge \mathrm{d}t^2 =$$

$$= \iint_D \sum_{i=1}^3 \Big(\frac{\partial P}{\partial x^i}\frac{\partial x^i}{\partial t^1}\frac{\partial x^1}{\partial t^2} - \frac{\partial P}{\partial x^i}\frac{\partial x^i}{\partial t^2}\frac{\partial x^1}{\partial t^1}\Big)\mathrm{d}t^1 \wedge \mathrm{d}t^2 =$$

$$= \iint_D \Big[\Big(\frac{\partial P}{\partial x^2}\frac{\partial x^2}{\partial t^1} + \frac{\partial P}{\partial x^3}\frac{\partial x^3}{\partial t^1}\Big)\frac{\partial x^1}{\partial t^2} - \Big(\frac{\partial P}{\partial x^2}\frac{\partial x^2}{\partial t^2} + \frac{\partial P}{\partial x^3}\frac{\partial x^3}{\partial t^2}\Big)\frac{\partial x^1}{\partial t^1}\Big]\mathrm{d}t^1 \wedge \mathrm{d}t^2 =$$

$$= \iint_D \left(\frac{\partial P}{\partial x^2}\begin{vmatrix} \frac{\partial x^2}{\partial t^1} & \frac{\partial x^2}{\partial t^2} \\ \frac{\partial x^1}{\partial t^1} & \frac{\partial x^1}{\partial t^2} \end{vmatrix} + \frac{\partial P}{\partial x^3}\begin{vmatrix} \frac{\partial x^3}{\partial t^1} & \frac{\partial x^3}{\partial t^2} \\ \frac{\partial x^1}{\partial t^1} & \frac{\partial x^1}{\partial t^2} \end{vmatrix}\right)\mathrm{d}t^1 \wedge \mathrm{d}t^2 =$$

$$=: \iint_S \Big(\frac{\partial P}{\partial x^2}\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial P}{\partial x^3}\mathrm{d}x^3 \wedge \mathrm{d}x^1\Big).$$

The colon here denotes equality by definition, and the exclamation point denotes a transition that uses the Green's formula already proved. The rest consists of identities.

Using the basic idea of the proof of formula (13.34′), we have thus verified directly (without invoking the relation $\varphi^*\mathrm{d} = \mathrm{d}\varphi^*$, but essentially proving it for the case under consideration) that formula (13.34) does indeed hold for a simple parametrized surface. We have carried out the reasoning formally only for the term $P\,\mathrm{d}x$, but it is clear that the same thing could also be done for the other two terms in the 1-form in the integrand on the left-hand side of (13.34).

13.3.4 The General Stokes Formula

Despite the differences in the external appearance of formulas (13.25), (13.30), and (13.34), their coordinate-free expressions (13.25″), (13.29), (13.30′), and (13.34′) turn out to be identical. This gives grounds for supposing that we have been dealing with particular manifestations of a general rule, which one can now easily guess.

Proposition 4. *Let S be an oriented piecewise smooth k-dimensional compact surface with boundary ∂S in the domain $G \subset \mathbb{R}^n$, in which a smooth $(k-1)$-form ω is defined.*

Then the following relation holds:

$$\boxed{\int\limits_S \mathrm{d}\omega = \int\limits_{\partial S} \omega\,,} \tag{13.35}$$

in which the orientation of the boundary ∂S is that induced by the orientation of S.

Proof. Formula (13.35) can obviously be proved by the same general computations (13.28) and (13.29) as Stokes' formula (13.34′) provided it holds for a standard k-dimensional interval $I^k = \{x = (x^1, \dots, x^k) \in \mathbb{R}^k \big| 0 \le x^i \le 1,\, i = 1, \dots, k\}$. Let us verify that (13.35) does indeed hold for I^k.

Since a $(k-1)$-form on I^k has the form $\omega = \sum_i a_i(x)\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^k$ (summation over $i = 1, \dots, k$, with the differential $\mathrm{d}x^i$ omitted), it suffices to prove (13.35) for each individual term. Let $\omega = a(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^k$. Then $\mathrm{d}\omega = (-1)^{i-1}\frac{\partial a}{\partial x^i}(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^k$. We now carry out the computation:

$$\int\limits_{I^k} \mathrm{d}\omega = \int\limits_{I^k} (-1)^{i-1}\frac{\partial a}{\partial x^i}(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k =$$

$$= (-1)^{i-1} \int\limits_{I^{k-1}} \mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^k \int\limits_0^1 \frac{\partial a}{\partial x^i}(x)\,\mathrm{d}x^i =$$

$$= (-1)^{i-1} \int\limits_{I^{k-1}} \big(a(x^1, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^k) -$$

$$- a(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots x^k)\big)\,\mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^k =$$

$$= (-1)^{i-1} \int\limits_{I^{k-1}} a(t^1, \dots, t^{i-1}, 1, t^i, \dots, t^{k-1})\mathrm{d}t^1 \cdots \mathrm{d}t^{k-1} +$$

$$+ (-1)^i \int\limits_{I^{k-1}} a(t^1, \dots, t^{i-1}, 0, t^i, \dots, t^{k-1})\,\mathrm{d}t^1 \cdots \mathrm{d}t^{k-1}\,.$$

Here I^{k-1} is the same as I^k in $\mathbb{R}^k$, only it is a $(k-1)$-dimensional interval in $\mathbb{R}^{k-1}$. In addition, we have relabeled the variables $x^1 = t^1, \dots, x^{i-1} = t^{i-1}$, $x^{i+1} = t^i, \dots, x^k = t^{k-1}$.

The mappings

$$I^{k-1} \ni t = (t^1, \dots, t^{k-1}) \longmapsto (t^1, \dots, t^{i-1}, 1, t^i, \dots, t^{k-1}) \in I^k ,$$
$$I^{k-1} \ni t = (t^1, \dots, t^{k-1}) \longmapsto (t^1, \dots, t^{i-1}, 0, t^i, \dots, t^{k-1}) \in I^k$$

are parametrizations of the upper and lower faces Γ_{i1} and Γ_{i0} of the interval I^k respectively orthogonal to the x^i axis. These coordinates define the same frame $\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_k$ orienting the faces and differing from the frame $\mathbf{e}_1, \dots, \mathbf{e}_k$ of $\mathbb{R}^k$ in the absence of $\mathbf{e}_i$. On Γ_{i1} the vector $\mathbf{e}_i$ is the exterior normal to I^k, as the vector $-\mathbf{e}_i$ is for the face Γ_{i0}. The frame $\mathbf{e}_i, \mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_k$ becomes the frame $\mathbf{e}_1, \dots, \mathbf{e}_k$ after $i-1$ interchanges of adjacent vectors, that is, the agreement or disagreement of the orientations of these frames is determined by the sign of $(-1)^{i-1}$. Thus, this parametrization defines an orientation on Γ_{i1} consistent with the orientation of I^k if taken with the corrective coefficient $(-1)^{i-1}$ (that is, not changing the orientation when i is odd, but changing it when i is even).

Analogous reasoning shows that for the face Γ_{i0} it is necessary to take a corrective coefficient $(-1)^i$ to the orientation defined by this parametrization of the face Γ_{i0}.

Thus, the last two integrals (together with the coefficients in front of them) can be interpreted respectively as the integrals of the form ω over the faces Γ_{i1} and Γ_{i0} of I^k with the orientation induced by the orientation of I^k.

We now remark that on each of the remaining faces of I^k one of the coordinates $x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^k$ is constant. Hence the differential corresponding to it is equal to zero on such a face. Thus, the form $\mathrm{d}\omega$ is identically equal to zero and its integral equals zero over all faces except Γ_{i0} and Γ_{i1}.

Hence we can interpret the sum of the integrals over these two faces as the integral of the form ω over the entire boundary ∂I^k of the interval I^k oriented in consistency with the orientation of the interval I^k itself.

The formula

$$\int\limits_{I_k} \mathrm{d}\omega = \int\limits_{\partial I^k} \omega ,$$

and along with it formula (13.35), is now proved. □

As one can see, formula (13.35) is a corollary of the Newton–Leibniz formula (fundamental theorem of calculus), Fubini's theorem, and a series of definitions of such concepts as surface, boundary of a surface, orientation, differential form, differentiation of a differential form, and transference of forms.

Formulas (13.25), (13.30), and (13.34), the formulas of Green, Gauss–Ostrogradskii, and Stokes respectively, are special cases of the general formula

(13.35). Moreover, if we interpret a function f defined on a closed interval $[a,b] \subset \mathbb{R}$ as a 0-form ω, and the integral of a 0-form over an oriented point as the value of the function at that point taken with the sign of the orientation of the point, then the Newton–Leibniz formula itself can be regarded as an elementary (but independent) version of (13.35). Consequently, the fundamental relation (13.35) holds in all dimensions $k \geq 1$.

Formula (13.35) is often called the *general Stokes formula.* As historical information, we quote here some lines from the preface of M. Spivak to his book (cited in the bibliography below):

> The first statement of the Theorem[9] appears as a postscript to a letter, dated July 2, 1850, from Sir William Thomson (Lord Kelvin) to Stokes. It appeared publicly as question 8 on the Smith's Prize Examination for 1854. This competitive examination, which was taken annually by the best mathematics students at Cambridge University, was set from 1849 to 1882 by Professor Stokes; by the time of his death the result was known universally as Stokes' theorem. At least three proofs were given by his contemporaries: Thomson published one, another appeared in Thomson and Tait's *Treatise on Natural Philosophy*, and Maxwell provided another in *Electricity and Magnetism*. Since this time the name of Stokes has been applied to much more general results, which have figured so prominently in the development of certain parts of mathematics that Stokes' theorem may be considered a case study in the value of generalization.

We note that the modern language of differential forms originates with Élie Cartan,[10] but the form (13.35) for the general Stokes' formula for surfaces in $\mathbb{R}^n$ seems to have been first proposed by Poincaré. For domains in n-dimensional space $\mathbb{R}^n$ Ostrogradskii already knew the formula, and Leibniz wrote down the first differential forms.

Thus it is not an accident that the general Stokes formula (13.35) is sometimes called the Newton–Leibniz–Green–Gauss–Ostrogradski–Stokes–Poincaré formula. One can conclude from what has been said that this is by no means its full name.

Let us use this formula to generalize the result of Example 2.

Example 5. Let us show that every smooth mapping $f : \overline{B} \to \overline{B}$ of a closed ball $\overline{B} \subset \mathbb{R}^m$ into itself has at least one fixed point.

Proof. If the mapping f had no fixed points, then, as in Example 2, one could construct a smooth mapping $\varphi : \overline{B} \to \partial\overline{B}$ that is the identity on the sphere $\partial\overline{B}$. In the domain $\mathbb{R}^m \setminus 0$, we consider the vector field $\frac{\mathbf{r}}{|\mathbf{r}|^m}$, where $\mathbf{r}$ is the radius-vector of the point $x = (x^1, \ldots, x^m) \in \mathbb{R}^m \setminus 0$, and the flux form

[9] The classical Stokes theorem (13.34) is meant.

[10] Élie Cartan (1869–1951) – outstanding French geometer.

$$\omega = \left\langle \frac{\mathbf{r}}{|\mathbf{r}|^m}, \mathbf{n} \right\rangle \Omega = \sum_{i=1}^{m} \frac{(-1)^{i-1} x^i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m}{\left((x^1)^2 + \cdots + (x^m)^2\right)^{m/2}}$$

corresponding to this field (see formula (13.19) of Sect. 13.2). The flux of such a field across the boundary of the ball $\overline{B} = \{x \in \mathbb{R} \,|\, |x| = 1\}$ in the direction of the outward normal to the sphere $\partial\overline{B}$ is obviously equal to the area of the sphere $\partial\overline{B}$, that is, $\int\limits_{\partial\overline{B}} \omega \neq 0$. But, as one can easily verify by direct computation, $\mathrm{d}\omega = 0$ in $\mathbb{R}^m \setminus 0$, from which, by using the general Stokes formula, as in Example 2, we find that

$$\int\limits_{\partial\overline{B}} \omega = \int\limits_{\partial\overline{B}} \varphi^*\omega = \int\limits_{\overline{B}} \mathrm{d}\varphi^*\omega = \int\limits_{\overline{B}} \varphi^*\mathrm{d}\omega = \int\limits_{\overline{B}} \varphi^* 0 = 0\,.$$

This contradiction finishes the proof. □

13.3.5 Problems and Exercises

1. a) Does Green's formula (13.25) change if we pass from the coordinate system x, y to the system y, x?

b) Does formula (13.25″) change in this case?

2. a) Prove that formula (13.25) remains valid if the functions P and Q are continuous in a closed square I, their partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous at interior points of I, and the double integrals exist, even if as improper integrals (13.25′).

b) Verify that if the boundary of a compact domain D consists of piecewise-smooth curves, then under assumptions analogous to those in a), formula (13.25) remains valid.

3. a) Verify the proof of (13.26′) in detail.

b) Show that if the boundary of a compact domain $D \subset \mathbb{R}^2$ consists of a finite number of smooth curves having only a finite number of points of inflection, then D is a simple domain with respect to any pair of coordinate axes.

c) Is it true that if the boundary of a plane domain consists of smooth curves, then one can choose the coordinate axes in $\mathbb{R}^2$ such that it is a simple domain relative to them?

4. a) Show that if the functions P and Q in Green's formula are such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then the area $\sigma(D)$ of the domain D can be found using the formula $\sigma(D) = \int\limits_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y$.

b) Explain the geometric meaning of the integral $\int\limits_{\gamma} y\,\mathrm{d}x$ over some (possibly nonclosed) curve in the plane with Cartesian coordinates x, y. Starting from this, give a new interpretation of the formula $\sigma(D) = -\int\limits_{\partial D} y\,\mathrm{d}x$.

c) As a check on the preceding formula, use it to find the area of the domain

$$D = \left\{ (x, y) \in \mathbb{R}^2 \middle| \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\} .$$

5. a) Let $x = x(t)$ be a diffeomorphism of the domain $D_t \subset \mathbb{R}^2_t$ onto the domain $D_x \subset \mathbb{R}^2_x$. Using the results of Problem 4 and the fact that a line integral is independent of the admissible change in the parametrization of the path, prove that

$$\int\limits_{D_x} \mathrm{d}x = \int\limits_{D_t} |x'(t)| \, \mathrm{d}t ,$$

where $\mathrm{d}x = \mathrm{d}x^1 \mathrm{d}x^2$, $\mathrm{d}t = \mathrm{d}t^1 \, \mathrm{d}t^2$, $|x'(t)| = \det x'(t)$.

b) From a) derive the formula

$$\int\limits_{D_x} f(x) \, \mathrm{d}x = \int\limits_{D_t} f\big(x(t)\big) |\det x'(t)| \, \mathrm{d}t$$

for change of variable in a double integral.

6. Let $f(x, y, t)$ be a smooth function satisfying the condition $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \neq 0$ in its domain of definition. Then for each fixed value of the parameter t the equation $f(x, y, t) = 0$ defines a curve γ_t in the plane $\mathbb{R}^2$. Then a family of curves $\{\gamma_t\}$ depending on the parameter t arises in the plane. A smooth curve $\Gamma \subset \mathbb{R}^2$ defined by parametric equations $x = x(t)$, $y = y(t)$, is the *envelope of the family of curves* $\{\gamma_t\}$ if the point $x(t_0), y(t_0)$ lies on the corresponding curve γ_{t_0} and the curves Γ and γ_{t_0} are tangent at that point, for every value of t_0 in the common domain of definition of $\{\gamma_t\}$ and the functions $x(t), y(t)$.

a) Assuming that x, y are Cartesian coordinates in the plane, show that the functions $x(t), y(t)$ that define the envelope must satisfy the system of equations

$$\begin{cases} f(x, y, t) = 0 , \\ \frac{\partial f}{\partial t}(x, y, t) = 0 , \end{cases}$$

and from the geometric point of view the envelope itself is the boundary of the projection (shadow) of the surface $f(x, y, t) = 0$ of $\mathbb{R}^3_{(x,y,t)}$ on the plane $\mathbb{R}^2_{(x,y)}$.

b) A family of lines $x \cos \alpha + y \sin \alpha - p(\alpha) = 0$ is given in the plane with Cartesian coordinates x and y. The role of the parameter is played here by the polar angle α. Give the geometric meaning of the quantity $p(\alpha)$, and find the envelope of this family if $p(\alpha) = c + a \cos \alpha + b \sin \alpha$, where a, b, and c are constants.

c) Desribe the accessible zone of a shell that can be fired from an adjustable cannon making any angle $\alpha \in [0, \pi/2]$ to the horizon.

d) Show that if the function $p(\alpha)$ of b) is 2π-periodic, then the corresponding envelope Γ is a closed curve.

e) Using Problem 4, show that the length L of the closed curve Γ obtained in d) can be found by the formula

$$L = \int_0^{2\pi} p(\alpha)\,\mathrm{d}\alpha\,.$$

(Assume that $p \in C^{(2)}$.)

f) Show also that the area σ of the region bounded by the closed curve Γ obtained in d) can be computed as

$$\sigma = \frac{1}{2}\int_0^{2\pi} (p^2 - \dot{p}^2)(\alpha)\,\mathrm{d}\alpha\,, \qquad \dot{p}(\alpha) = \frac{\mathrm{d}p}{\mathrm{d}\alpha}(\alpha)\,.$$

7. Consider the integral $\int\limits_{\gamma} \frac{\cos(\mathbf{r},\mathbf{n})}{r}\,\mathrm{d}s$, in which γ is a smooth curve in $\mathbb{R}^2$, $\mathbf{r}$ is the radius-vector of the point $(x,y) \in \gamma$, $r = |\mathbf{r}| = \sqrt{x^2+y^2}$, $\mathbf{n}$ is the unit normal vector to γ at (x,y) varying continuously along γ, and $\mathrm{d}s$ is arc length on the curve. This integral is called *Gauss' integral.*

a) Write Gauss' integral in the form of a flux $\int\limits_{\gamma} \langle \mathbf{V},\mathbf{n}\rangle\,\mathrm{d}s$ of the plane vector field $\mathbf{V}$ across the curve γ.

b) Show that in Cartesian coordinates x and y Gauss' integral has the form $\pm\int\limits_{\gamma} \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2+y^2}$ familiar to us from Example 1 of Sect. 13.1, where the choice of sign is determined by the choice of the field of normals $\mathbf{n}$.

c) Compute Gauss' integral for a closed curve γ that encircles the origin once and for a curve γ bounding a domain that does not contain the origin.

d) Show that $\frac{\cos(\mathbf{r},\mathbf{n})}{r}\,\mathrm{d}s = \mathrm{d}\varphi$, where φ is the polar angle of the radius-vector $\mathbf{r}$, and give the geometric meaning of the value of Gauss' integral for a closed curve and for an arbitrary curve $\gamma \subset \mathbb{R}^2$.

8. In deriving the Gauss–Ostrogradskii formula we assumed that D is a simple domain and the functions P, Q, R belong to $C^{(1)}(\overline{D},\mathbb{R})$. Show by improving the reasoning that formula (13.30) holds if D is a compact domain with piecewise smooth boundary, $P,Q,R \in C(\overline{D},\mathbb{R})$, $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z} \in C(D,\mathbb{R})$, and the triple integral converges, even if it is an improper integral.

9. a) If the functions P, Q, and R in formula (13.30) are such that $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1$, then the volume $V(D)$ of the domain D can be found by the formula

$$V(D) = \iint\limits_{\partial D} P\,\mathrm{d}y\wedge\mathrm{d}z + Q\,\mathrm{d}z\wedge\mathrm{d}x + R\,\mathrm{d}x\wedge\mathrm{d}y\,.$$

b) Let $f(x,t)$ be a smooth function of the variables $x \in D_X \subset \mathbb{R}^n_x$, $t \in D_t \subset \mathbb{R}^n_t$ and $\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x^1},\dots,\frac{\partial f}{\partial x^n}\right) \neq 0$. Write the system of equations that must be satisfied by the $(n-1)$-dimensional surface in $\mathbb{R}^n_x$ that is the envelope of the family of surfaces $\{S_t\}$ defined by the conditions $f(x,t) = 0$, $t \in D_t$ (see Problem 6).

c) Choosing a point on the unit sphere as the parameter t, exhibit a fmaily of planes in $\mathbb{R}^3$ depending on the parameter t whose envelope is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

d) Show that if a closed surface S is the envelope of a family of planes

$$\cos\alpha_1(t)x + \cos\alpha(t)y + \cos\alpha_3(t)z - p(t) = 0 ,$$

where $\alpha_1, \alpha_2, \alpha_3$ are the angles formed by the normal to the plane and the coordinate axes and the parameter t is a variable point of the unit sphere $S^2 \subset \mathbb{R}^3$, then the area σ of the surface S can be found by the formula $\sigma = \int\limits_{S^2} p(t)\,d\sigma$.

e) Show that the volume of the body bounded by the surface S considered in d) can be found by the formula $V = \frac{1}{3}\int\limits_{S} p(t)\,d\sigma$.

f) Test the formula given in e) by finding the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

g) What does the n-dimensional analogue of the formulas in d) and e) look like?

10. a) Using the Gauss–Ostrogradskii formula, verify that the flux of the field $\mathbf{r}/r^3$ (where $\mathbf{r}$ is the radius-vector of the point $x \in \mathbb{R}^3$ and $r = |\mathbf{r}|$) across a smooth surface S enclosing the origin and homeomorphic to a sphere equals the flux of the same field across an arbitrarily small sphere $|x| = \varepsilon$.

b) Show that the flux in a) is 4π.

c) Interpret Gauss' integral $\int\limits_{S} \frac{\cos(\mathbf{r},\mathbf{n})}{r}\,ds$ in $\mathbb{R}^3$ as the flux of the field $\mathbf{r}/r^3$ across the surface S.

d) Compute Gauss' integral over the boundary of a compact domain $D \subset \mathbb{R}^3$, considering both the case when D contains the origin in its interior and the case when the origin lies outside D.

e) Comparing Problems 7 and 10a)–d), give an n-dimensional version of Gauss' integral and the corresponding vector field. Give an n-dimensional statement of problems a)–d) and verify it.

11. a) Show that a closed rigid surface $S \subset \mathbb{R}^3$ remains in equilibrium under the action of a uniformly distributed pressure on it. (By the principles of statics the problem reduces to verifying the equalities $\iint\limits_{S} \mathbf{n}\,d\sigma = 0$, $\iint\limits_{S} [\mathbf{r},\mathbf{n}]\,d\sigma = 0$, where $\mathbf{n}$ is a unit normal vector, $\mathbf{r}$ is the radius-vector, and $[\mathbf{r},\mathbf{n}]$ is the vector product of $\mathbf{r}$ and $\mathbf{n}$.)

b) A solid body of volume V is completely immersed in a liquid having specific gravity 1. Show that the complete static effect of the pressure of the liquid on the body reduces to a single force $\mathbf{F}$ of magnitude V directed vertically upward and attached to the center of mass C of the solid domain occupied by the body.

12. Let $\Gamma : I^k \to D$ be a smooth (not necessarily homeomorphic) mapping of an interval $I^k \subset \mathbb{R}^k$ into a domain D of $\mathbb{R}^n$, in which a k-form ω is defined. By analogy with the one-dimensional case, we shall call a mapping Γ a *k-cell* or *k-pathand* by definition set $\int\limits_{\Gamma} \omega = \int\limits_{I^k} \Gamma^*\omega$. Study the proof of the general Stokes formula and verify that it holds not only for k-dimensional surfaces but also for k-cells.

13. Using the generalized Stokes formula, prove by induction the formula for change of variable in a multiple integral (the idea of the proof is shown in Problem 5a)).

14. *Integration by parts in a multiple integral.*

Let D be a bounded domain in $\mathbb{R}^m$ with a regular (smooth or piecewise smooth) boundary ∂D oriented by the outward unit normal $\mathbf{n} = (n^1, \dots, n^m)$.

Let f, g be smooth functions in $\overline{D}$.

a) Show that

$$\int_D \partial_i f \, \mathrm{d}v = \int_{\partial D} f n^i \, \mathrm{d}\sigma .$$

b) Prove the following formula for integration by parts:

$$\int_D (\partial_i f) g \, \mathrm{d}v = \int_{\partial D} f g n^i \, \mathrm{d}\sigma - \int_D f(\partial_i g) \, \mathrm{d}v .$$

14 Elements of Vector Analysis and Field Theory

14.1 The Differential Operations of Vector Analysis

14.1.1 Scalar and Vector Fields

In field theory we consider functions $x \mapsto T(x)$ that assign to each point x of a given domain D a special object $T(x)$ called a *tensor*. If such a function is defined in a domain D, we say that a *tensor field* is defined in D. We do not intend to give the definition of a tensor at this point: that concept will be studied in algebra and differential geometry. We shall say only that numerical functions $D \ni x \mapsto f(x) \in \mathbb{R}$ and vector-valued functions $\mathbb{R}^n \supset D \ni x \mapsto V(x) \in T\mathbb{R}^n_x \approx \mathbb{R}^n$ are special cases of tensor fields and are called *scalar fields* and *vector fields* respectively in D (we have used this terminology earlier).

A differential p-form ω in D is a function $\mathbb{R}^n \supset D \ni x \mapsto \omega(x) \in \mathcal{L}\big((\mathbb{R}^n)^p, \mathbb{R}\big)$ which can be called a *field of forms* of degree p in D. This also is a special case of a tensor field.

At present we are primarily interested in scalar and vector fields in domains of the oriented Euclidean space $\mathbb{R}^n$. These fields play a major role in many applications of analysis in natural science.

14.1.2 Vector Fields and Forms in $\mathbb{R}^3$

We recall that in the Euclidean vector space $\mathbb{R}^3$ with inner product $\langle\, ,\rangle$ there is a correspondence between linear functionals $A : \mathbb{R}^3 \to \mathbb{R}$ and vectors $\mathbf{A} \in \mathbb{R}^3$ consisting of the following: Each such functional has the form $A(\boldsymbol{\xi}) = \langle \mathbf{A}, \boldsymbol{\xi}\rangle$, where $\mathbf{A}$ is a completely definite vector in $\mathbb{R}^3$.

If the space is also oriented, each skew-symmetric bilinear functional $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ can be uniquely written in the form $B(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = (\mathbf{B}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$, where $\mathbf{B}$ is a completely definite vector in $\mathbb{R}^3$ and $(\mathbf{B}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$, as always, is the scalar triple product of the vectors $\mathbf{B}$, $\boldsymbol{\xi}_1$, and $\boldsymbol{\xi}_2$, or what is the same, the value of the volume element on these vectors. Thus, in the oriented Euclidean vector space $\mathbb{R}^3$ one can associate with each vector a linear or bilinear form, and defining the linear or bilinear form is equivalent to defining the corresponding vector in $\mathbb{R}^3$.

If there is an inner product in $\mathbb{R}^3$, it also arises naturally in each tangent space $T\mathbb{R}^3_x$ consisting of the vectors attached to the point $\mathbb{R}^3$, and the orientation of $\mathbb{R}^3$ orients each space $T\mathbb{R}^3_x$.

Hence defining a 1-form $\omega^1(x)$ or a 2-form $\omega^2(x)$ in $T\mathbb{R}^3_x$ under the conditions just listed is equivalent to defining some vector $\mathbf{A}(x) \in T\mathbb{R}^3_x$ corresponding to the form $\omega^1(x)$ or a vector $\mathbf{B}(x) \in T\mathbb{R}^3_x$ corresponding to the form $\omega^2(x)$.

Consequently, defining a 1-form ω^1 or a 2-form ω^2 in a domain D of the oriented Euclidean space $\mathbb{R}^3$ is equivalent to defining the vector field $\mathbf{A}$ or $\mathbf{B}$ in D corresponding to the form.

In explicit form, this correspondence amounts to the following:

$$\omega^1_{\mathbf{A}}(x)(\boldsymbol{\xi}) = \langle \mathbf{A}(x), \boldsymbol{\xi} \rangle \,, \tag{14.1}$$

$$\omega^2_{\mathbf{B}}(x)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \big(\mathbf{B}(x), \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\big) \,, \tag{14.2}$$

where $\mathbf{A}(x)$, $\mathbf{B}(x)$, $\boldsymbol{\xi}$, $\boldsymbol{\xi}_1$, and $\boldsymbol{\xi}_2$ belong to TD_x.

Here we see the work form $\omega^1 = \omega^1_{\mathbf{A}}$ of the vector field $\mathbf{A}$ and the flux form $\omega^2 = \omega^2_{\mathbf{B}}$ of the vector field $\mathbf{B}$, which are already familiar to us.

To a scalar field $f : D \to \mathbb{R}$, we can assign a 0-form and a 3-form in D as follows:

$$\omega^0_f = f \,, \tag{14.3}$$

$$\omega^3_f = f \, \mathrm{d}V \,, \tag{14.4}$$

where $\mathrm{d}V$ is the volume element in the oriented Euclidean space $\mathbb{R}^3$.

In view of the correspondences (14.1)–(14.4), definite operations on vector and scalar fields correspond to operations on forms. This observation, as we shall soon verify, is very useful technically.

Proposition 1. *To a linear combination of forms of the same degree there corresponds a linear combination of the vector and scalar fields corresponding to them.*

Proof. Proposition 1 is of course obvious. However, let us write out the full proof, as an example, for 1-forms:

$$\alpha_1 \omega^1_{\mathbf{A}_1} + \alpha_2 \omega^1_{\mathbf{A}_2} = \alpha_1 \langle \mathbf{A}_1, \cdot \rangle + \alpha \langle \mathbf{A}_2, \cdot \rangle = \\ = \langle \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2, \cdot \rangle = \omega^2_{\alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2} \,. \quad \square$$

It is clear from the proof that α_1 and α_2 can be regarded as functions (not necessarily constant) in the domain D in which the forms and fields are defined.

As an abbreviation, let us agree to use, along with the symbols $\langle \, , \, \rangle$ and $[\, , \,]$ for the inner product and the vector product of vectors $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^3$, the alternative notation $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$ wherever convenient.

Proposition 2. *If* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}_1$, *and* $\mathbf{B}_1$ *are vector fields in the oriented Euclidean space* $\mathbb{R}^3$, *then*

$$\omega^1_{\mathbf{A}_1} \wedge \omega^1_{\mathbf{A}_2} = \omega^2_{\mathbf{A}_1 \times \mathbf{A}_2} , \tag{14.5}$$

$$\omega^1_{\mathbf{A}} \wedge \omega^2_{\mathbf{B}} = \omega^3_{\mathbf{A} \cdot \mathbf{B}} . \tag{14.6}$$

In other words, the vector product $\mathbf{A}_1 \times \mathbf{A}_2$ of fields $\mathbf{A}_1$ and $\mathbf{A}_2$ that generate 1-forms corresponds to the exterior product of the 1-forms they generate, since it generates the 2-form that results from the product.

In the same sense the inner product of the vector fields $\mathbf{A}$ and $\mathbf{B}$ that generate a 1-form $\omega^1_{\mathbf{A}}$ and a 2-form $\omega^2_{\mathbf{B}}$ corresponds to the exterior product of these forms.

Proof. To prove these assertions, fix an orthornormal basis in $\mathbb{R}^3$ and the Cartesian coordinates x^1, x^2, x^3 corresponding to it.

In Cartesian coordinates

$$\omega^1_{\mathbf{A}}(x)(\boldsymbol{\xi}) = \mathbf{A}(x) \cdot \boldsymbol{\xi} = \sum_{i=1}^{3} A^i(x)\xi^i = \sum_{i=1}^{3} A^i(x)\, \mathrm{d}x^i(\boldsymbol{\xi}) ,$$

that is,

$$\omega^1_{\mathbf{A}} = A^1\, \mathrm{d}x^1 + A^2\, \mathrm{d}x^2 + A^3\, \mathrm{d}x^3 , \tag{14.7}$$

and

$$\omega^2_{\mathbf{B}}(x)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \begin{vmatrix} B^1(x) & B^2(x) & B^3(x) \\ \xi^1_1 & \xi^2_1 & \xi^3_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 \end{vmatrix} =$$
$$= \big(B^1(x)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B^2\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B^3(x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\big)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) ,$$

that is,

$$\omega^2_{\mathbf{B}} = B^1\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B^2\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B^3\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 . \tag{14.8}$$

Therefore in Cartesian coordinates, taking account of expressions (14.7) and (14.8), we obtain

$$\omega^1_{\mathbf{A}_1} \wedge \omega^1_{\mathbf{A}_2} = (A^1_1\, \mathrm{d}x^1 + A^2_1\, \mathrm{d}x^2 + A^3_1\, \mathrm{d}x^3) \wedge (A^1_2\, \mathrm{d}x^1 + A^2_2\, \mathrm{d}x^2 + A^3_2\, \mathrm{d}x^3) =$$
$$= (A^2_1 A^3_2 - A^3_1 A^2_2)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + (A^3_1 A^1_2 - A^1_1 A^3_2)\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 +$$
$$+ (A^1_1 A^2_2 - A^2_1 A^1_2)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \omega^2_{\mathbf{B}} ,$$

where $\mathbf{B} = \mathbf{A}_1 \times \mathbf{A}_2$.

Coordinates were used in this proof only to make it easier to find the vector $\mathbf{B}$ of the corresponding 2-form. The equality (14.5) itself, of course, is independent of the coordinate system.

Similarly, multiplying Eqs. (14.7) and (14.8), we obtain

$$\omega^1_{\mathbf{A}} \wedge \omega^2_{\mathbf{B}} = (A^1 B^1 + A^2 B^2 + A^3 B^3)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \omega^3_{\rho} .$$

In Cartesian coordinates $dx^1 \wedge dx^2 \wedge dx^3$ is the volume element in $\mathbb{R}^3$, and the sum of the pairwise products of the coordinates of the vectors $\mathbf{A}$ and $\mathbf{B}$, which appears in parentheses just before the 3-form, is the inner product of these vectors at the corresponding points of the domain, from which it follows that $\rho(x) = \mathbf{A}(x) \cdot \mathbf{B}(x)$. □

14.1.3 The Differential Operators grad, curl, div, and ∇

Definition 1. To the exterior differentiation of 0-forms (functions), 1-forms, and 2-forms in oriented Euclidean space $\mathbb{R}^3$ there correspond respectively the operations of finding the *gradient* (grad) of a scalar field and the *curl* and *divergence* (div) of a vector field. These operations are defined by the relations

$$d\omega_f^0 =: \omega_{\operatorname{grad} f}^1 , \tag{14.9}$$

$$d\omega_{\mathbf{A}}^1 =: \omega_{\operatorname{curl} \mathbf{A}}^2 , \tag{14.10}$$

$$d\omega_{\mathbf{B}}^1 =: \omega_{\operatorname{div} \mathbf{B}}^3 . \tag{14.11}$$

By virtue of the correspondence between forms and scalar and vector fields in $\mathbb{R}^3$ established by Eqs. (14.1)–(14.4), relations (14.9)–(14.11) are unambiguous definitions of the operations grad, curl, and div, performed on scalar and vector fields respectively. These operations, the *operators of field theory* as they are called, correspond to the single operation of exterior differentiation of forms, but applied to forms of different degree.

Let us give right away the explicit form of these operators in Cartesian coordinates x^1, x^2, x^3 in $\mathbb{R}^3$.

As we have explained, in this case

$$\omega_f^0 = f , \tag{14.3'}$$

$$\omega_{\mathbf{A}}^1 = A^1\, dx^1 + A^2\, dx^2 + A^3\, dx^3 , \tag{14.7'}$$

$$\omega_{\mathbf{B}}^2 = B^1\, dx^2 \wedge dx^3 + B^2\, dx^3 \wedge dx^1 + B^3\, dx^1 \wedge dx^2 , \tag{14.8'}$$

$$\omega_\rho^3 = \rho\, dx^1 \wedge dx^2 \wedge dx^3 . \tag{14.4'}$$

Since

$$\omega_{\operatorname{grad} f}^1 := d\omega_f^0 = df = \frac{\partial f}{\partial x^1}\, dx^1 + \frac{\partial f}{\partial x^2}\, dx^2 + \frac{\partial f}{\partial x^3}\, dx^3 ,$$

it follows from (14.7′) that in these coordinates

$$\operatorname{grad} f = \mathbf{e}_1 \frac{\partial f}{\partial x^1} + \mathbf{e}_2 \frac{\partial f}{\partial x^2} + \mathbf{e}_3 \frac{\partial f}{\partial x^3} , \tag{14.9'}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a fixed orthonormal basis of $\mathbb{R}^3$.

Since

$$\omega^2_{\operatorname{curl}\mathbf{A}} := \mathrm{d}\omega^1_{\mathbf{A}} = \mathrm{d}(A^1\,\mathrm{d}x^1 + A^2\,\mathrm{d}x^2 + A^3\,\mathrm{d}x^3) = \\ = \Big(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}\Big)\,\mathrm{d}x^2\wedge \mathrm{d}x^3 + \Big(\frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}\Big)\,\mathrm{d}x^3\wedge \mathrm{d}x^1 + \\ + \Big(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\Big)\,\mathrm{d}x^1\wedge \mathrm{d}x^2\,,$$

it follows from (14.8′) that in Cartesian coordinates

$$\operatorname{curl}\mathbf{A} = \mathbf{e}_1\Big(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}\Big) + \mathbf{e}_2\Big(\frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}\Big) + \mathbf{e}_3\Big(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\Big)\,. \qquad (14.10')$$

As an aid to memory this last relation is often written in symbolic form as

$$\operatorname{curl}\mathbf{A} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ A^1 & A^2 & A^3 \end{vmatrix}\,. \qquad (14.10'')$$

Next, since

$$\omega^3_{\operatorname{div}\mathbf{B}} := \mathrm{d}\omega^2_{\mathbf{B}} = \mathrm{d}(B^1\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + B^2\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + B^3\,\mathrm{d}x^1\wedge\mathrm{d}x^2) = \\ = \Big(\frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3}\Big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\,,$$

it follows from (14.4′) that in Cartesian coordinates

$$\operatorname{div}\mathbf{B} = \frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3}\,. \qquad (14.11')$$

One can see from the formulas (14.9′), (14.10′), and (14.11′) just obtained that grad, curl, and div are linear differential operations (operators). The grad operator is defined on differentiable scalar fields and assigns vector fields to the scalar fields. The curl operator is also vector-valued, but is defined on differentiable vector fields. The div operator is defined on differentiable vector fields and assigns scalar fields to them.

We note that in other coordinates these operators will have expressions that are in general different from those obtained above in Cartesian coordinates. We shall discuss this point in Subsect. 14.1.5 below.

We remark also that the vector field curl $\mathbf{A}$ is sometimes called the *rotation* of $\mathbf{A}$ and written rot $\mathbf{A}$.

As an example of the use of these operators we write out the famous[1] system of equations of Maxwell,[2] which describe the state of the components of an electromagnetic field as functions of a point $x = (x^1, x^2, x^3)$ in space and time t.

Example 1. (The Maxwell equations for an electromagnetic field in a vacuum.)

$$\begin{aligned} &1.\ \operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0}\,. && 2.\ \operatorname{div} \mathbf{B} = 0\,. \\ &3.\ \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,. && 4.\ \operatorname{curl} \mathbf{B} = \frac{\mathbf{j}}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,. \end{aligned} \tag{14.12}$$

Here $\rho(x,t)$ is the electric charge density (the quantity of charge per unit volume), $\mathbf{j}(x,t)$ is the electrical current density vector (the rate at which charge is flowing across a unit area), $\mathbf{E}(x,t)$ and $\mathbf{B}(x,t)$ are the electric and magnetic field intensities respectively, and ε_0 and c are dimensioning constants (and in fact c is the speed of light in a vacuum).

In mathematical and especially in physical literature, along with the operators grad, curl, and div, wide use is made of the symbolic differential operator nabla proposed by Hamilton (the *Hamilton operator*)[3]

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x^1} + \mathbf{e}_2 \frac{\partial}{\partial x^2} + \mathbf{e}_3 \frac{\partial}{\partial x^3}\,, \tag{14.13}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ and x^1, x^2, x^3 are the corresponding Cartesian coordinates.

By definition, applying the operator ∇ to a scalar field f (that is, to a function), gives the vector field

[1] On this subject the famous American physicist and mathematician R. Feynman (1918–1988) writes, with his characteristic acerbity, "From a long view of the history of mankind – seen from, say, ten thousand years from now – there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade." Richard R. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics: Mainly Electromagnetism and Matter*, Reading, MA: Addison-Wesley, 1964.

[2] J.C. Maxwell (1831–1879) – outstanding Scottish physicist; he created the mathematical theory of the electromagnetic field and is also famous for his researach in the kinetic theory of gases, optics and mechanics.

[3] W.R. Hamilton (1805–1865) – famous Irish mathematician and specialist in mechanics; he stated the variational principle (Hamilton's principle) and constructed a phenomenological theory of optic phenomena; he was the creator of the theory of quaternions and the founder of vector analysis (in fact, the term "vector" is due to him).

$$\nabla f = \mathbf{e}_1 \frac{\partial f}{\partial x^1} + \mathbf{e}_2 \frac{\partial f}{\partial x^2} + \mathbf{e}_3 \frac{\partial f}{\partial x^3} ,$$

which coincides with the field (14.9′), that is, the nabla operator is simply the grad operator written in a different notation.

Using, however, the vector form in which ∇ is written, Hamilton proposed a system of formal operations with it that imitates the corresponding algebraic operations with vectors.

Before we illustrate these operations, we note that in dealing with ∇ one must adhere to the same principles and cautionary rules as in dealing with the usual differentiation operator $D = \frac{\mathrm{d}}{\mathrm{d}x}$. For example, $\varphi D f$ equals $\varphi \frac{\mathrm{d}f}{\mathrm{d}x}$ and not $\frac{\mathrm{d}}{\mathrm{d}x}(\varphi f)$ or $f \frac{\mathrm{d}\varphi}{\mathrm{d}x}$. Thus, the operator operates on whatever is placed to the right of it; left multiplication in this case plays the role of a coefficient, that is, φD is the new differential operator $\varphi \frac{\mathrm{d}}{\mathrm{d}x}$, not the function $\frac{\mathrm{d}\varphi}{\mathrm{d}x}$. Moreover, $D^2 = D \cdot D$, that is, $D^2 f = D(Df) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x} f\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f$.

If we now, following Hamilton, deal with ∇ as if it were a vector field defined in Cartesian coordinates, then, comparing relations (14.13), (14.9′), (14.10″), and (14.11′), we obtain

$$\operatorname{grad} f = \nabla f \tag{14.14}$$

$$\operatorname{curl} \mathbf{A} = \nabla \times \mathbf{A} , \tag{14.15}$$

$$\operatorname{div} \mathbf{B} = \nabla \cdot \mathbf{B} . \tag{14.16}$$

In this way the operators grad, curl, and div, can be written in terms of the Hamilton operator and the vector operations in $\mathbb{R}^3$.

Example 2. Only the curl and div operators occurred in writing out the Maxwell equations (14.12). Using the principles for dealing with $\nabla = \operatorname{grad}$, we rewrite the Maxwell equations as follows, to compensate for the absence of grad in them:

$$\begin{aligned} &1.\ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} . && 2.\ \nabla \cdot \mathbf{B} = 0 . \\ &3.\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} . && 4.\ \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} . \end{aligned} \tag{14.12′}$$

14.1.4 Some Differential Formulas of Vector Analysis

In the oriented Euclidean space $\mathbb{R}^3$ we have established the connection (14.1)–(14.4) between forms on the one hand and vector and scalar fields on the other. This connection enabled us to associate corresponding operators on fields with exterior differentiation (see formulas (14.5), (14.6), and (14.9)–(14.11)).

This correspondence can be used to obtain a number of basic differential formulas of vector analysis.

For example, the following relations hold:

$$\operatorname{curl}(f\mathbf{A}) = f\operatorname{curl}\mathbf{A} - \mathbf{A}\times\operatorname{grad} f\,, \tag{14.17}$$

$$\operatorname{div}(f\mathbf{A}) = \mathbf{A}\cdot\operatorname{grad} f + f\operatorname{div}\mathbf{A}\,, \tag{14.18}$$

$$\operatorname{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\operatorname{curl}\mathbf{A} - \mathbf{A}\cdot\operatorname{curl}\mathbf{B}\,. \tag{14.19}$$

Proof. We shall verify this last equality:

$$\omega^3_{\operatorname{div}\mathbf{A}\times\mathbf{B}} = \mathrm{d}\omega^2_{\mathbf{A}\times\mathbf{B}} = \mathrm{d}(\omega^1_{\mathbf{A}}\wedge\omega^1_{\mathbf{B}}) = \mathrm{d}\omega^1_{\mathbf{A}}\wedge\omega^1_{\mathbf{B}} - \omega^1_{\mathbf{A}}\wedge\mathrm{d}\omega^1_{\mathbf{B}} =$$
$$= \omega^2_{\operatorname{curl}\mathbf{A}}\wedge\omega^1_{\mathbf{B}} - \omega^1_{\mathbf{A}}\wedge\omega^2_{\operatorname{curl}\mathbf{B}} = \omega^3_{\mathbf{B}\cdot\operatorname{curl}\mathbf{A}} - \omega^3_{\mathbf{A}\cdot\operatorname{curl}\mathbf{B}} = \omega^3_{\mathbf{B}\cdot\operatorname{curl}\mathbf{A}-\mathbf{A}\cdot\operatorname{curl}\mathbf{B}}\,.$$

The first two relations are verified similarly. Of course, the verification of all these equalities can also be carried out by direct differentiation in coordinates. □

If we take account of the relation $\mathrm{d}^2\omega = 0$ for any form ω, we can also assert that the following equalities hold:

$$\operatorname{curl}\operatorname{grad} f = \mathbf{0}\,, \tag{14.20}$$

$$\operatorname{div}\operatorname{curl}\mathbf{A} = 0\,. \tag{14.21}$$

Proof. Indeed:

$$\omega^2_{\operatorname{curl}\operatorname{grad} f} = \mathrm{d}\omega^1_{\operatorname{grad} f} = \mathrm{d}(\mathrm{d}\omega^0_f) = \mathrm{d}^2\omega^0_f = 0\,,$$
$$\omega^3_{\operatorname{div}\operatorname{curl}\mathbf{A}} = \mathrm{d}\omega^2_{\operatorname{curl}\mathbf{A}} = \mathrm{d}(\mathrm{d}\omega^1_{\mathbf{A}}) = \mathrm{d}^2\omega^1_{\mathbf{A}} = 0\,. \ \square$$

In formulas (14.17)–(14.19) the operators grad, curl, and div are applied once, while (14.20) and (14.21) involve the second-order operators obtained by successive execution of two of the three original operations. Besides the rules given in (14.20) and (14.21), one can also consider other combinations of these operators:

$$\operatorname{grad}\ \operatorname{div}\mathbf{A}, \quad \operatorname{curl}\operatorname{curl}\mathbf{A}, \quad \operatorname{div}\operatorname{grad} f\,. \tag{14.22}$$

The operator div grad is applied, as one can see, to a scalar field. This operator is denoted Δ (Delta) and is called the *Laplace operator*[4] or *Laplacian*.

It follows from (14.9′) and (14.11′) that in Cartesian coordinates

$$\Delta f = \frac{\partial^2 f}{\partial (x^1)^2} + \frac{\partial^2 f}{\partial (x^2)^2} + \frac{\partial^2 f}{\partial (x^3)^2}\,. \tag{14.23}$$

[4] P.S. Laplace (1749–1827) – famous French astronomer, mathematician, and physicist; he made fundamental contributions to the development of celestial mechanics, the mathematical theory of probability, and experimental and mathematical physics.

Since the operator Δ acts on numerical functions, it can be applied componentwise to the coordinates of vector fields $\mathbf{A} = \mathbf{e}_1 A^1 + \mathbf{e}_2 A^2 + \mathbf{e}_3 A^3$, where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are an orthonormal basis in $\mathbb{R}^3$. In that case

$$\Delta\mathbf{A} = \mathbf{e}_1 \Delta A^1 + \mathbf{e}_2 \Delta A^2 + \mathbf{e}_3 \Delta A^3 .$$

Taking account of this last equality, we can write the following relation for the triple of second-order operators (14.22):

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \Delta\mathbf{A} , \tag{14.24}$$

whose proof we shall not take the time to present (see Problem 2 below). The equality (14.24) can serve as the definition of $\Delta\mathbf{A}$ in any coordinate system, not necessarily orthogonal.

Using the language of vector algebra and formulas (14.14)–(14.16), we can write all the second-order operators (14.20)–(14.22) in terms of the Hamilton operator ∇:

$$\begin{aligned}
\operatorname{curl}\ \operatorname{grad} f &= \nabla \times \nabla f = 0 , \\
\operatorname{div}\ \operatorname{curl}\mathbf{A} &= \nabla \cdot (\nabla \times \mathbf{A}) = 0 , \\
\operatorname{grad}\ \operatorname{div}\mathbf{A} &= \nabla(\nabla \cdot \mathbf{A}) , \\
\operatorname{curl}\ \operatorname{curl}\mathbf{A} &= \nabla \times (\nabla \times \mathbf{A}) , \\
\operatorname{div}\ \operatorname{grad} f &= \nabla \cdot \nabla f .
\end{aligned}$$

From the point of view of vector algebra the vanishing of the first two of these operators seems completely natural.

The last equality means that the following relation holds between the Hamilton operator ∇ and the Laplacian Δ:

$$\Delta = \nabla^2 .$$

14.1.5 *Vector Operations in Curvilinear Coordinates

a. Just as, for example, the sphere $x^2 + y^2 + z^2 = a^2$ has a particularly simple equation $R = a$ in spherical coordinates, vector fields $x \mapsto \mathbf{A}(x)$ in $\mathbb{R}^3$ (or $\mathbb{R}^n$) often assume a simpler expression in a coordinate system that is not Cartesian. For that reason we now wish to find explicit formulas from which one can find grad, curl, and div in a rather extensive class of curvilinear coordinates.

But first it is necessary to be precise as to what is meant by the coordinate expression for a field $\mathbf{A}$ in a curvilinear coordinate system.

We begin with two introductory examples of a descriptive character.

Example 3. Suppose we have a fixed Cartesian coordinate system x^1, x^2 in the Euclidean plane $\mathbb{R}^2$. When we say that a vector field $(A^1, A^2)(x)$ is defined in $\mathbb{R}^2$, we mean that some vector $\mathbf{A}(x) \in T\mathbb{R}^2_x$ is connected with each point $x = (x^1, x^2) \in \mathbb{R}^2$, and in the basis of $T\mathbb{R}^2_x$ consisting of the unit vectors $\mathbf{e}_1(x)$, $\mathbf{e}_2(x)$ in the coordinate directions we have the expansion $\mathbf{A}(x) = A^1(x)\mathbf{e}_1(x) + A^2(x)\mathbf{e}_2(x)$ (see Fig. 14.1). In this case the basis $\{\mathbf{e}_1(x), \mathbf{e}_2(x)\}$ of $T\mathbb{R}^2_x$ is essentially independent of x.

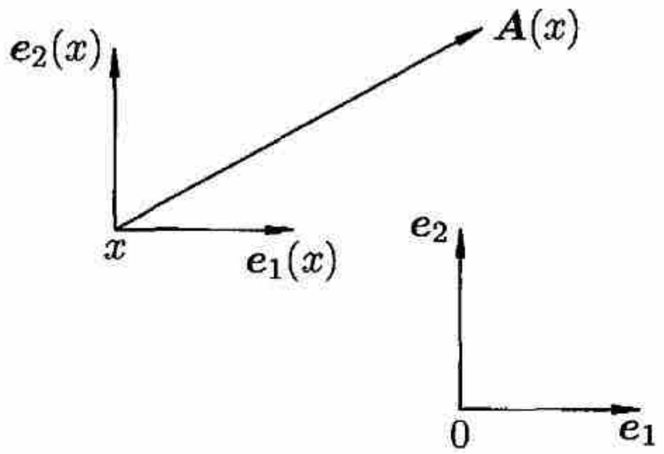

Fig. 14.1.

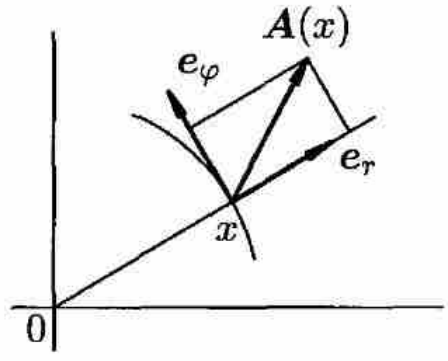

Fig. 14.2.

Example 4. In the case when polar coordinates (r, φ) are defined in the same plane $\mathbb{R}^2$, at each point $x \in \mathbb{R}^2 \setminus 0$ one can also attach unit vectors $\mathbf{e}_1(x) = \mathbf{e}_r(x)$, $\mathbf{e}_2 = \mathbf{e}_\varphi(x)$ (Fig. 14.2) in the coordinate directions. They also form a basis in $T\mathbb{R}^2_x$ with respect to which one can expand the vector $\mathbf{A}(x)$ of the field $\mathbf{A}$ attached to x: $\mathbf{A}(x) = A^1(x)\mathbf{e}_1(x) + A^2(x)\mathbf{e}_2(x)$. It is then natural to regard the ordered pair of functions $(A^1, A^2)(x)$ as the expression for the field $\mathbf{A}$ in polar coordinates.

Thus, if $(A^1, A^2)(x) \equiv (1, 0)$, this is a field of unit vectors in $\mathbb{R}^2$ pointing radially away from the center 0.

The field $(A^1, A^2)(x) \equiv (0, 1)$ can be obtained from the preceding field by rotating each vector in it counterclockwise by the angle $\pi/2$.

These are not constant fields in $\mathbb{R}^2$, although the components of their coordinate representation are constant. The point is that the basis in which the expansion is taken varies synchronously with the vector of the field in a transition from one point to another.

It is clear that the components of the coordinate representation of these fields in Cartesian coordinates would not be constant at all. On the other hand, a truly constant field (consisting of a vector translated parallel to itself to all points of the plane) which does have constant components in a Cartesian coordinate system, would have variable components in polar coordinates.

b. After these introductory considerations, let us consider more formally the problem of defining vector fields in curvilinear coordinate systems.

We recall first of all that a curvilinear coordinate system t^1, t^2, t^3 in a domain $D \subset \mathbb{R}^3$ is a diffeomorphism $\varphi : D_t \to D$ of a domain D_t in the Euclidean parameter space $\mathbb{R}^3_t$ onto the domain D, as a result of which each point $x = \varphi(t) \in D$ acquires the Cartesian coordinates t^1, t^2, t^3 of the corresponding point $t \in D_t$.

Since φ is a diffeomorphism, the tangent mapping $\varphi'(t) : T\mathbb{R}^3_t \to T\mathbb{T}^3_{x=\varphi(t)}$ is a vector-space isomorphism. To the canonical basis $\boldsymbol{\xi}_1(t) = (1,0,0)$, $\boldsymbol{\xi}_2(t) = (0,1,0)$, $\boldsymbol{\xi}_3(t) = (0,0,1)$ of $T\mathbb{R}^3_t$ corresponds the basis of $T\mathbb{R}^3_{x=\varphi(t)}$ consisting of the vectors $\boldsymbol{\xi}_i(x) = \varphi'(t)\boldsymbol{\xi}_i(t) = \frac{\partial\varphi(t)}{\partial t^i}$, $i = 1,2,3$, giving the coordinate directions. To the expansion $\mathbf{A}(x) = \alpha_1\boldsymbol{\xi}_1(x) + \alpha_2\boldsymbol{\xi}_2(x) + \alpha_3\boldsymbol{\xi}_3(x)$ of any vector $\mathbf{A}(x) \in T\mathbb{R}^3_x$ in this basis there corresponds the same expansion $\mathbf{A}(t) = \alpha_1\boldsymbol{\xi}_1(t) + \alpha_2\boldsymbol{\xi}_2(t) + \alpha_3\boldsymbol{\xi}_3(t)$ (with the same components $\alpha_1, \alpha_2, \alpha_3$!) of the vector $\mathbf{A}(t) = (\varphi')^{-1}\mathbf{A}(x)$ in the canonical basis $\boldsymbol{\xi}_1(t), \boldsymbol{\xi}_2(t), \boldsymbol{\xi}_3(t)$ in $T\mathbb{R}^3_t$. In the absense of a Euclidean structure in $\mathbb{R}^3$, the numbers $\alpha_1, \alpha_2, \alpha_3$ would be the most natural coordinate expression for the vector $\mathbf{A}(x)$ connected with this curvilinear coordinate system.

c. However, adopting such a coordinate representation would not be quite consistent with what we agreed to in Example 4. The point is that the basis $\boldsymbol{\xi}_1(x), \boldsymbol{\xi}_2(x), \boldsymbol{\xi}_3(x)$ in $T\mathbb{R}^3_x$ corresponding to the canonical basis $\boldsymbol{\xi}_1(t), \boldsymbol{\xi}_2(t), \boldsymbol{\xi}_3(t)$ in $T\mathbb{R}^3_t$, although it consists of vectors in the coordinate directions, is not at all required to consist of *unit vectors* in those directions, that is, in general $\langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_i \rangle(x) \neq 1$.

We shall now take account of this circumstance which results from the presence of a Euclidean structure in $\mathbb{R}^3$ and consequently in each vector space $T\mathbb{R}^3_x$ also.

Because of the isomorphism $\varphi'(t) : T\mathbb{R}^3_t \to T\mathbb{R}^3_{x=\varphi(t)}$ we can transfer the Euclidean structure of $T\mathbb{R}^3_x$ into $T\mathbb{R}^3_t$ by setting $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle := \langle \varphi'\boldsymbol{\tau}_1, \varphi'\boldsymbol{\tau}_2 \rangle$ for every pair of vectors $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in T\mathbb{R}^3_t$. In particular, we obtain from this the following expression for the square of the length of a vector:

$$\langle \boldsymbol{\tau}, \boldsymbol{\tau} \rangle = \langle \varphi'(t)\boldsymbol{\tau}, \varphi'(t)\boldsymbol{\tau} \rangle = \Big\langle \frac{\partial\varphi(t)}{\partial t^i}\tau^i, \frac{\partial\varphi(t)}{\partial t^j}\tau^j \Big\rangle =$$
$$= \Big\langle \frac{\partial\varphi}{\partial t^i}, \frac{\partial\varphi}{\partial t^j} \Big\rangle(t)\tau^i\tau^j = \langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_j \rangle(t)\tau^i\tau^j = g_{ij}(t)\mathrm{d}t^i(\boldsymbol{\tau})\,\mathrm{d}t^j(\boldsymbol{\tau})\ .$$

The quadratic form

$$\mathrm{d}s^2 = g_{ij}(t)\,\mathrm{d}t^i\,\mathrm{d}t^j \tag{14.25}$$

whose coefficients are the pairwise inner products of the vectors in the canonical basis determines the inner product on $T\mathbb{R}^3_t$ completely. If such a form is defined at each point of a domain $D_t \subset \mathbb{R}^3_t$, then, as is known from geometry, one says that a *Riemannian metric* is defined in this domain. A Riemannian metric makes it possible to introduce a Euclidean structure in each tangent space $T\mathbb{R}^3_t$ $(t \in D_t)$ within the context of rectilinear coordinates t^1, t^2, t^3 in

$\mathbb{R}^3_t$, corresponding to the "curved" embedding $\varphi : D_t \to D$ of the domain D_t in the Euclidean space $\mathbb{R}^3$.

If the vectors $\boldsymbol{\xi}_i(x) = \varphi'(t)\boldsymbol{\xi}_i(t) = \frac{\partial\varphi}{\partial t^i}(t)$, $i = 1,2,3$, are orthogonal in $T\mathbb{R}^3_x$, then $g_{ij}(t) = 0$ for $i \neq j$. This means that we are dealing with a *triorthogonal coordinate grid*. In terms of the space $T\mathbb{R}^3_t$ it means that the vectors $\boldsymbol{\xi}_i(t)$, $i = 1,2,3$, in the canonical basis are mutually orthogonal in the sense of the inner product in $T\mathbb{R}^3_t$ defined by the quadratic form (14.25). In what follows, for the sake of simplicity, we shall consider only triorthogonal curvilinear coordinate systems. For them, as has been noted, the quadratic form (14.25) has the following special form:

$$\mathrm{d}s^2 = E_1(t)(\mathrm{d}t^1)^2 + E_2(t)(\mathrm{d}t^2)^2 + E_3(t)(\mathrm{d}t^3)^2\ , \tag{14.26}$$

where $E_i(t) = g_{ii}(t)$, $i = 1,2,3$.

Example 5. In Cartesian coordinates (x,y,z), cylindrical coordinates (r,φ,z), and spherical coordinates (R,φ,θ) on Euclidean space $\mathbb{R}^3$ the quadratic form (14.25) has the respective forms

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = && (14.26')\\
&= \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2 = && (14.26'')\\
&= \mathrm{d}R^2 + R^2\cos^2\theta\,\mathrm{d}\varphi^2 + R^2\,\mathrm{d}\theta^2\ . && (14.26''')
\end{aligned}$$

Thus, each of these coordinate systems is a triorthogonal system in its domain of definition.

The vectors $\boldsymbol{\xi}_1(t), \boldsymbol{\xi}_2(t), \boldsymbol{\xi}_3(t)$ of the canonical basis $(1,0,0),(0,1,0),(0,0,1)$ in $T\mathbb{R}^3_t$, like the vectors $\boldsymbol{\xi}_i(x) \in T\mathbb{R}^3_x$ corresponding to them, have the following norm[5]: $|\boldsymbol{\xi}_i| = \sqrt{g_{ii}}$. Hence the unit vectors (in the sense of the square-norm of a vector) in the coordinate directions have the following coordinate representation for the triorthogonal system (14.26):

$$\mathbf{e}_1(t) = \Big(\frac{1}{\sqrt{E_1}},0,0\Big),\ \mathbf{e}_2(t) = \Big(0,\frac{1}{\sqrt{E_2}},0\Big),\ \mathbf{e}_3(t) = \Big(0,0,\frac{1}{\sqrt{E_3}}\Big)\ . \tag{14.27}$$

Example 6. It follows from formulas (14.27) and the results of Example 5 that for Cartesian, cylindrical, and spherical coordinates, the triples of unit vectors along the coordinate directions have respectively the following forms:

$$\begin{aligned}
&\mathbf{e}_x = (1,0,0)\ , \quad \mathbf{e}_y = (0,1,0)\ , && \mathbf{e}_z = (0,0,1)\ ; && (14.27')\\
&\mathbf{e}_r = (1,0,0)\ , \quad \mathbf{e}_\varphi = \Big(0,\frac{1}{r},0\Big)\ , && \mathbf{e}_z = (0,0,1)\ ; && (14.27'')\\
&\mathbf{e}_R = (1,0,0)\ , \quad \mathbf{e}_\varphi = \Big(0,\frac{1}{R\cos\theta},0\Big)\ , && \mathbf{e}_\theta = \Big(0,0,\frac{1}{R}\Big)\ ; && (14.27''')
\end{aligned}$$

[5] In the triorthogonal system (14.26) we have $|\boldsymbol{\xi}_i| = \sqrt{E_i} = H_i$, $i = 1,2,3$. The quantities H_1, H_2, H_3 are usually called the *Lamé coefficients* or *Lamé parameters*. G. Lamé (1795–1870) – French engineer, mathematician, and physicist.

Examples 3 and 4 considered above assumed that the vector of the field was expanded in a basis consisting of *unit vectors* along the coordinate directions. Hence the vector $\mathbf{A}(t) \in T\mathbb{R}^3_t$ corresponding to the vector $\mathbf{A}(x) \in T\mathbb{R}^3_x$ of the field should be expanded in the basis $\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)$ consisting of unit vectors in the coordinate directions, rather than in the canonical basis $\boldsymbol{\xi}_1(t), \boldsymbol{\xi}_2(t), \boldsymbol{\xi}_3(t)$.

Thus, abstracting from the original space $\mathbb{R}^3$, one can assume that a Riemannian metric (14.25) or (14.26) and a vector field $t \mapsto \mathbf{A}(t)$ are defined in the domain $D_t \subset \mathbb{R}^3_t$ and that the coordinate representation $(A^1, A^2, A^3)(t)$ of $\mathbf{A}(t)$ at each point $t \in D_t$ is obtained from the expansion of the vector $\mathbf{A}(t) = A^i(t)\mathbf{e}_i(t)$ of the field corresponding to this point with respect to unit vectors along the coordinate axes.

d. Let us now investigate forms. Under the diffeomorphism $\varphi : D_t \to D$ every form in D automatically transfers to the domain D_t. This transfer, as we know, occurs at each point $x \in D$ from the space $T\mathbb{R}^3_x$ into the corresponding space $T\mathbb{R}^3_t$. Since we have transferred the Euclidean structure into $T\mathbb{R}^3_t$ from $T\mathbb{R}^3_x$, it follows from the definition of the transfer of vectors and forms that, for example, to a given form $\omega^1_{\mathbf{A}}(x) = \langle \mathbf{A}(x), \cdot \rangle$ defined in $T\mathbb{R}^3_x$ there corresponds exactly the same kind of form $\omega^1_{\mathbf{A}}(t) = \langle \mathbf{A}(t), \cdot \rangle$ in $T\mathbb{R}^3_t$, where $\mathbf{A}(x) = \varphi'(t)\mathbf{A}(t)$. The same can be said of forms of the type $\omega^2_{\mathbf{B}}$ and ω^3_ρ, to say nothing of forms ω^0_f – that is, functions.

After these clarifications, the rest of our study can be confined to the domain $D_t \subset \mathbb{R}^3_t$, abstracting from the original space $\mathbb{R}^3$ and assuming that a Riemannian metric (14.25) is defined in D_t and that scalar fields f, ρ and vector fields $\mathbf{A}$, $\mathbf{B}$ are defined in D_t along with the forms ω^0_f, $\omega^1_{\mathbf{A}}$, $\omega^2_{\mathbf{B}}$, ω^3_ρ, which are defined at each point $t \in D_t$ in accordance with the Euclidean structure on $T\mathbb{R}^3_t$ defined by the Riemannian metric.

Example 7. The volume element $\mathrm{d}V$ in curvilinear coordinates t^1, t^2, t^3, as we know, has the form

$$\mathrm{d}V = \sqrt{\det g_{ij}}(t)\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 \wedge \mathrm{d}t^3 .$$

For a triorthogonal system

$$\mathrm{d}V = \sqrt{E_1 E_2 E_3}(t)\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 \wedge \mathrm{d}t^3 . \tag{14.28}$$

In particular, in Cartesian, cylindrical, and spherical coordinates, respectively, we obtain

$$\mathrm{d}V = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \tag{14.28$'$}$$

$$= r\,\mathrm{d}r \wedge \mathrm{d}\varphi \wedge \mathrm{d}z = \tag{14.28$''$}$$

$$= R^2 \cos\theta\,\mathrm{d}R \wedge \mathrm{d}\varphi \wedge \mathrm{d}\theta . \tag{14.28$'''$}$$

What has just been said enables us to write the form $\omega^3_\rho = \rho\,\mathrm{d}V$ in different curvilinear coordinate systems.

e. Our main problem (now easily solvable) is, knowing the expansion $\mathbf{A}(t) = A^i(t)\mathbf{e}_i(t)$ for a vector $\mathbf{A}(t) \in T\mathbb{R}^3_t$ with respect to the unit vectors $\mathbf{e}_i(t) \in T\mathbb{R}^3_t$, $i = 1, 2, 3$, of the triorthogonal coordinate system determined by the Riemannian metric (14.26), to find the expansion of the forms $\omega^1_{\mathbf{A}}(t)$ and $\omega^2_{\mathbf{B}}(t)$ in terms of the canonical 1-forms $\mathrm{d}t^i$ and the canonical 2-forms $\mathrm{d}t^i \wedge \mathrm{d}t^j$ respectively.

Since all the reasoning applies at every given point t, we shall abbreviate the notation by suppressing the letter t that shows that the vectors and forms are attached to the tangent space at t.

Thus, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a basis in $T\mathbb{R}^3_t$ consisting of the unit vectors (14.27) along the coordinate directions, and $\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3$ is the expansion of $\mathbf{A} \in T\mathbb{R}^3_t$ in that basis.

We remark first of all that formula (14.27) implies that

$$\mathrm{d}t^j(\mathbf{e}_i) = \frac{1}{\sqrt{E_i}}\delta^i_j\,, \quad \text{where} \quad \delta^i_j = \begin{cases} 0\,, & \text{if } i \neq j\,, \\ 1\,, & \text{if } i = j\,, \end{cases} \tag{14.29}$$

$$\mathrm{d}t^i \wedge \mathrm{d}t^j(\mathbf{e}_k, \mathbf{e}_l) = \frac{1}{\sqrt{E_iE_j}}\delta^{ij}_{kl}\,, \quad \text{where} \quad \delta^{ij}_{kl} = \begin{cases} 0, & \text{if } (i,j) \neq (k,l)\,, \\ 1\,, & \text{if } (i,j) = (k,l)\,. \end{cases} \tag{14.30}$$

f. Thus, if $\omega^1_{\mathbf{A}} := \langle \mathbf{A}, \cdot \rangle = a_1\mathrm{d}t^1 + a_2\mathrm{d}t^2 + a_3\mathrm{d}t^3$, then on the one hand

$$\omega^1_{\mathbf{A}}(\mathbf{e}_i) = \langle \mathbf{A}, \mathbf{e}_i \rangle = A^i\,,$$

and on the other hand, as one can see from (14.29),

$$\omega^1_{\mathbf{A}}(\mathbf{e}_i) = (a_1\,\mathrm{d}t^1 + a_2\,\mathrm{d}t^2 + a_3\,\mathrm{d}t^3)(\mathbf{e}_i) = a_i \cdot \frac{1}{\sqrt{E_i}}\,.$$

Consequently, $a_i = A^i\sqrt{E_i}$, and we have found the expansion

$$\omega^1_{\mathbf{A}} = A^1\sqrt{E_1}\,\mathrm{d}t^1 + A^2\sqrt{E_2}\,\mathrm{d}t^2 + A^3\sqrt{E_3}\,\mathrm{d}t^3 \tag{14.31}$$

for the form $\omega^1_{\mathbf{A}}$ corresponding to the expansion $\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3$ of the vector $\mathbf{A}$.

Example 8. Since in Cartesian, spherical, and cylindrical coordinates we have respectively

$$\begin{aligned} \mathbf{A} &= A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z = \\ &= A_r\mathbf{e}_r + A_\varphi\mathbf{e}_\varphi + A_z\mathbf{e}_z \\ &= A_R\mathbf{e}_R + A_\varphi\mathbf{e}_\varphi + A_\theta\mathbf{e}_\theta\,, \end{aligned}$$

as follows from the results of Example 6,

$$\omega^1_{\mathbf{A}} = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_z\,\mathrm{d}z = \tag{14.31'}$$
$$= A_r\,\mathrm{d}r + A_\varphi r\,\mathrm{d}\varphi + A_z\,\mathrm{d}z = \tag{14.31''}$$
$$= A_R\,\mathrm{d}R + A_\varphi R\cos\varphi\,\mathrm{d}\varphi + A_\theta R\,\mathrm{d}\theta\,. \tag{14.31'''}$$

g. Now let $\mathbf{B} = B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3$ and $\omega^2_{\mathbf{B}} = b_1\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + b_2\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + b_3\,\mathrm{d}t^1 \wedge \mathrm{d}t^2$. Then, on the one hand,

$$\omega^2_{\mathbf{B}}(\mathbf{e}_2,\mathbf{e}_3) := \mathrm{d}V(\mathbf{B},\mathbf{e}_2,\mathbf{e}_3) = \\ = \sum_{i=1}^{3} B^i\,\mathrm{d}V(\mathbf{e}_i,\mathbf{e}_2,\mathbf{e}_3) = B^1\cdot(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = B^1\,,$$

where $\mathrm{d}V$ is the volume element in $T\mathbb{R}^3_t$ (see (14.28) and (14.27)).

On the other hand, by (14.30) we obtain

$$\omega^2_{\mathbf{B}}(\mathbf{e}_2,\mathbf{e}_3) = (b_1\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + b_2\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + b_3\,\mathrm{d}t^1 \wedge \mathrm{d}t^2)(\mathbf{e}_2,\mathbf{e}_3) = \\ = b_1\,\mathrm{d}t^2 \wedge \mathrm{d}t^3(\mathbf{e}_2,\mathbf{e}_3) = \frac{b_1}{\sqrt{E_2E_3}}\,.$$

Comparing these results, we conclude that $b_1 = B^1\sqrt{E_2E_3}$. Similarly, we verify that $b_2 = B^2\sqrt{E_1E_3}$ and $b_3 = B^3\sqrt{E_1E_2}$.

Thus we have found the representation

$$\omega^2_{\mathbf{B}} = B^1\sqrt{E_2E_3}\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + B^2\sqrt{E_3E_1}\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + B^3\sqrt{E_1E_2}\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 = \\ = \sqrt{E_1E_2E_3}\Big(\frac{B_1}{\sqrt{E_1}}\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + \frac{B_2}{\sqrt{E_2}}\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + \frac{B_3}{\sqrt{E_3}}\,\mathrm{d}t^1 \wedge \mathrm{d}t^2\Big) \tag{14.32}$$

of the form $\omega^2_{\mathbf{B}}$ corresponding to the vector $\mathbf{B} = B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3$.

Example 9. Using the notation introduced in Example 8 and formulas (14.26′), (14.26″) and (14.26‴), we obtain in Cartesian, cylindrical, and spherical coordinates respectively

$$\omega^2_{\mathbf{B}} = B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y = \tag{14.32'}$$
$$= B_r r\,\mathrm{d}\varphi \wedge \mathrm{d}z + B_\varphi\,\mathrm{d}z \wedge \mathrm{d}r + B_z r\,\mathrm{d}r \wedge \mathrm{d}\varphi = \tag{14.32''}$$
$$= B_R R^2\cos\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta + B_\varphi R\,\mathrm{d}\theta \wedge \mathrm{d}R + B_\theta R\cos\theta\,\mathrm{d}R \wedge \mathrm{d}\varphi\,. \tag{14.32'''}$$

h. We add further that on the basis of (14.28) we can write

$$\omega^3_\rho = \rho\sqrt{E_1E_2E_3}\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 \wedge \mathrm{d}t^3\,. \tag{14.33}$$

Example 10. In particular, for Cartesian, cylindrical, and spherical coordinates respectively, formula (14.33) has the following forms:

$$\omega^3_\rho = \rho\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \tag{14.33'}$$

$$= \rho r\,\mathrm{d}r \wedge \mathrm{d}\varphi \wedge \mathrm{d}z = \tag{14.33''}$$

$$= \rho R^2 \cos\theta\,\mathrm{d}R \wedge \mathrm{d}\varphi \wedge \mathrm{d}\theta\,. \tag{14.33'''}$$

Now that we have obtained formulas (14.31)–(14.33), it is easy to find the coordinate representation of the operators grad, curl, and div in a triorthogonal curvilinear coordinate system using Definitions (14.9)–(14.11).

Let grad $f = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3$. Using the definitions, we write

$$\omega^1_{\mathrm{grad}\,f} := \mathrm{d}\omega^0_f := \mathrm{d}f := \frac{\partial f}{\partial t^1}\mathrm{d}t^1 + \frac{\partial f}{\partial t^2}\mathrm{d}t^2 + \frac{\partial f}{\partial t^3}\mathrm{d}t^3\,.$$

From this, using formula (14.31), we conclude that

$$\mathrm{grad}\,f = \frac{1}{\sqrt{E_1}}\frac{\partial f}{\partial t^1}\mathbf{e}_1 + \frac{1}{\sqrt{E_2}}\frac{\partial f}{\partial t^2}\mathbf{e}_2 + \frac{1}{\sqrt{E_3}}\frac{\partial f}{\partial t^3}\mathbf{e}_3\,. \tag{14.34}$$

Example 11. In Cartesian, polar, and spherical coordinates respectively,

$$\mathrm{grad}\,f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z = \tag{14.34'}$$

$$= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial f}{\partial z}\mathbf{e}_z = \tag{14.34''}$$

$$= \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R\cos\theta}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi + \frac{1}{R^2}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta\,. \tag{14.34'''}$$

Suppose given a field $\mathbf{A}(t) = \big(A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3\big)(t)$. Let us find the coordinates B^1, B^2, B^3 of the field $\mathrm{curl}\,\mathbf{A}(t) = \mathbf{B}(t) = (B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3)(t)$.

Based on the definition (14.10) and formula (14.31), we obtain

$$\omega^2_{\mathrm{curl}\,\mathbf{A}} := \mathrm{d}\omega^1_{\mathbf{A}} = \mathrm{d}\big(A^1\sqrt{E_1}\mathrm{d}t^1 + A^2\sqrt{E_2}\mathrm{d}t^2 + A^3\sqrt{E_3}\mathrm{d}t^3\big) =$$
$$= \Big(\frac{\partial A^3\sqrt{E_3}}{\partial t^2} - \frac{\partial A^2\sqrt{E_2}}{\partial t^3}\Big)\mathrm{d}t^2 \wedge \mathrm{d}t^3 +$$
$$+ \Big(\frac{\partial A^1\sqrt{E_1}}{\partial t^3} - \frac{\partial A^3\sqrt{E_3}}{\partial t^1}\Big)\mathrm{d}t^3 \wedge \mathrm{d}t^1 + \Big(\frac{\partial A^2\sqrt{E_2}}{\partial t^1} - \frac{\partial A^1\sqrt{E_1}}{\partial t^2}\Big)\mathrm{d}t^1 \wedge \mathrm{d}t^2\,.$$

On the basis of (14.32) we now conclude that

$$B^1 = \frac{1}{\sqrt{E_2E_3}}\left(\frac{\partial A^3\sqrt{E_3}}{\partial t^2} - \frac{\partial A^2\sqrt{E_2}}{\partial t^3}\right),$$
$$B^2 = \frac{1}{\sqrt{E_3E_1}}\left(\frac{\partial A^1\sqrt{E_1}}{\partial t^3} - \frac{\partial A^3\sqrt{E_3}}{\partial t^1}\right),$$
$$B^3 = \frac{1}{\sqrt{E_1E_2}}\left(\frac{\partial A^2\sqrt{E_2}}{\partial t^1} - \frac{\partial A^1\sqrt{E_1}}{\partial t^2}\right),$$

that is,

$$\operatorname{curl}\mathbf{A} = \frac{1}{\sqrt{E_1E_2E_3}}\begin{vmatrix} \sqrt{E_1}\mathbf{e}_1 & \sqrt{E_2}\mathbf{e}_2 & \sqrt{E_3}\mathbf{e}_3 \\ \dfrac{\partial}{\partial t^1} & \dfrac{\partial}{\partial t^2} & \dfrac{\partial}{\partial t^3} \\ \sqrt{E_1}A^1 & \sqrt{E_2}A^2 & \sqrt{E_3}A^3 \end{vmatrix} . \tag{14.35}$$

Example 12. In Cartesian, cylindrical, and spherical coordinates respectively

$$\operatorname{curl}\mathbf{A} = \Big(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\Big)\mathbf{e}_x + \Big(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\Big)\mathbf{e}_y + \Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big)\mathbf{e}_z = \tag{14.35'}$$

$$= \frac{1}{r}\Big(\frac{\partial A_z}{\partial \varphi} - \frac{\partial rA_\varphi}{\partial z}\Big)\mathbf{e}_r + \Big(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\Big)\mathbf{e}_\varphi + \frac{1}{r}\Big(\frac{\partial rA_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi}\Big)\mathbf{e}_z = \tag{14.35''}$$

$$= \frac{1}{R\cos\theta}\Big(\frac{\partial A_\theta}{\partial \varphi} - \frac{\partial A_\varphi \cos\theta}{\partial \theta}\Big)\mathbf{e}_R + \frac{1}{R}\Big(\frac{\partial A_R}{\partial \theta} - \frac{\partial RA_\theta}{\partial R}\Big)\mathbf{e}_\varphi + \\ + \frac{1}{R}\Big(\frac{\partial RA_\varphi}{\partial R} - \frac{1}{\cos\theta}\frac{\partial A_R}{\partial \varphi}\Big)\mathbf{e}_\theta . \tag{14.35'''}$$

i. Now suppose given a field $\mathbf{B}(t) = (B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3)(t)$. Let us find an expression for $\operatorname{div}\mathbf{B}$.

Starting from the definition (14.11) and formula (14.32), we obtain

$$\omega_{\operatorname{div}\mathbf{B}} := \mathrm{d}\omega^2_{\mathbf{B}} = \mathrm{d}\big(B^1\sqrt{E_2E_3}\mathrm{d}t^2 \wedge \mathrm{d}t^3 + \\ + B^2\sqrt{E_3E_1}\mathrm{d}t^3 \wedge \mathrm{d}t^1 + B^3\sqrt{E_1E_2}\mathrm{d}t^1 \wedge \mathrm{d}t^2 = \\ = \Big(\frac{\partial\sqrt{E_2E_3}B^1}{\partial t^1} + \frac{\partial\sqrt{E_3E_1}B^2}{\partial t^2} + \frac{\partial\sqrt{E_1E_2}B^3}{\partial t^3}\Big)\mathrm{d}t^1 \wedge \mathrm{d}t^2 \wedge \mathrm{d}t^3 .$$

On the basis of formula (14.33) we now conclude that

$$\operatorname{div}\mathbf{B} = \frac{1}{\sqrt{E_1E_2E_3}}\Big(\frac{\partial\sqrt{E_2E_3}B^1}{\partial t^1} + \frac{\partial\sqrt{E_3E_1}B^2}{\partial t^2} + \frac{\partial\sqrt{E_1E_2}B^3}{\partial t^3}\Big) . \tag{14.36}$$

In Cartesian, cylindrical, and spherical coordinates respectively, we obtain

$$\operatorname{div}\mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \tag{14.36'}$$

$$= \frac{1}{r}\Big(\frac{\partial rB_r}{\partial r} + \frac{\partial B_\varphi}{\partial \varphi}\Big) + \frac{\partial B_z}{\partial z} = \tag{14.36''}$$

$$= \frac{1}{R^2\cos\theta}\Big(\frac{\partial R^2\cos\theta B_R}{\partial R} + \frac{\partial RB_\varphi}{\partial \varphi} + \frac{\partial R\cos\theta B_\theta}{\partial \theta}\Big) . \tag{14.36'''}$$

j. Relations (14.34) and (14.36) can be used to obtain an expression for the Laplacian $\Delta = \operatorname{div}\operatorname{grad}$ in an arbitrary triorthogonal coordinate system:

$$\Delta f = \operatorname{div}\operatorname{grad} f = \operatorname{div}\Big(\frac{1}{\sqrt{E_1}}\frac{\partial f}{\partial t^1}\mathbf{e}_1 + \frac{1}{\sqrt{E_2}}\frac{\partial f}{\partial t^2}\mathbf{e}_2 + \frac{1}{\sqrt{E_3}}\frac{\partial f}{\partial t^3}\mathbf{e}_3\Big) =$$
$$= \frac{1}{\sqrt{E_1E_2E_3}}\bigg(\frac{\partial}{\partial t^1}\Big(\sqrt{\frac{E_2E_3}{E_1}}\frac{\partial f}{\partial t^1}\Big) +$$
$$+\frac{\partial}{\partial t^2}\Big(\sqrt{\frac{E_3E_1}{E_2}}\frac{\partial f}{\partial t^2}\Big) + \frac{\partial}{\partial t^3}\Big(\sqrt{\frac{E_1E_2}{E_3}}\frac{\partial f}{\partial t^3}\Big)\bigg)\,. \quad (14.37)$$

Example 13. In particular, for Cartesian, cylindirical, and spherical coordinates, we obtain respectively

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \quad (14.37')$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial f}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} = \quad (14.37'')$$
$$= \frac{1}{R^2}\frac{\partial}{\partial R}\Big(R^2\frac{\partial f}{\partial R}\Big) + \frac{1}{R^2\cos^2\theta}\frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{R^2\cos\theta}\frac{\partial}{\partial \theta}\Big(\cos\theta\frac{\partial f}{\partial \theta}\Big)\,. \quad (14.37''')$$

14.1.6 Problems and Exercises

1. The operators grad, curl, and div and the algebraic operations.
Verify the following relations:

for grad:

a) $\nabla(f+g) = \nabla f + \nabla g$,
b) $\nabla(f\cdot g) = f\nabla g + g\nabla f$,
c) $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$.
d) $\nabla\big(\frac{1}{2}\mathbf{A}^2\big) = (\mathbf{A}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{A})$;

for curl:

e) $\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + \nabla f\times\mathbf{A}$,
f) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B}$;

for div:

g) $\nabla\cdot(f\mathbf{A}) = \nabla f\cdot\mathbf{A} + f\nabla\cdot\mathbf{A}$,
h) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

and rewrite them in the symbols grad, curl, and div.

(Hints. $\mathbf{A}\cdot\nabla = A^1\frac{\partial}{\partial x^1} + A^2\frac{\partial}{\partial x^2} + A^3\frac{\partial}{\partial x^3}$; $\mathbf{B}\cdot\nabla \neq \nabla\cdot\mathbf{B}$; $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.)

2. a) Write the operators (14.20)–(14.22) in Cartesian coordinates.

b) Verify relations (14.20) and (14.21) by direct computation.

c) Verify formula (14.24) in Cartesian coordinates.

d) Write formula (14.24) in terms of ∇ and prove it, using the formulas of vector algebra.

3. From the system of Maxwell equations in Example 2 deduce that $\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}$.

4. a) Exhibit the Lamé parameters H_1, H_2, H_3 of Cartesian, cylindrical, and spherical coordinates in $\mathbb{R}^3$.

b) Rewrite formulas (14.28), (14.34)–(14.37), using the Lamé parameters.

5. Write the field $\mathbf{A} = \operatorname{grad} \frac{1}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$ in

a) Cartesian coordinates x, y, z;

b) cylindrical coordinates;

c) spherical coordinates.

d) Find $\operatorname{curl} \mathbf{A}$ and $\operatorname{div} \mathbf{A}$.

6. In cylindrical coordinates (r, φ, z) the function f has the form $\ln \frac{1}{r}$. Write the field $\mathbf{A} = \operatorname{grad} f$ in

a) Cartesian coordinates;

b) cylindrical coordinates;

c) spherical coordinates.

d) Find $\operatorname{curl} \mathbf{A}$ and $\operatorname{div} \mathbf{A}$.

7. Write the formula for transformation of coordinates in a fixed tangent space $T\mathbb{R}^3_p$, $p \in \mathbb{R}^3$, when passing from Cartesian coordinates in $\mathbb{R}^3$ to

a) cylindrical coordinates;

b) spherical coordinates;

c) an arbitrary triorthogonal curvilinear coordinate system.

d) Applying the formulas obtained in c) and formulas (14.34)–(14.37), verify directly that the vector fields $\operatorname{grad} f$, $\operatorname{curl} \mathbf{A}$, and the quantities $\operatorname{div} \mathbf{A}$ and Δf are invariant relative to the choice of the coordinate system in which they are computed.

8. The space $\mathbb{R}^3$, being a rigid body, revolves about a certain axis with constant angular velocity ω. Let $\mathbf{v}$ be the field of linear velocities of the points at a fixed instant of time.

a) Write the field $\mathbf{v}$ in the corresponding cylindrical coordinates.

b) Find $\operatorname{curl} \mathbf{v}$.

c) Indicate how the field $\operatorname{curl} \mathbf{v}$ is directed relative to the axis of rotation.

d) Verify that $|\operatorname{curl} \mathbf{v}| = 2\omega$ at each point of space.

e) Interpret the geometric meaning of $\operatorname{curl} \mathbf{v}$ and the geometric meaning of the constancy of this vector at all points of space for the situation in d).

14.2 The Integral Formulas of Field Theory

14.2.1 The Classical Integral Formulas in Vector Notation

a. Vector Notation for the Forms $\omega^1_{\mathbf{A}}$ and $\omega^2_{\mathbf{B}}$ In the preceding chapter we noted (see Sect. 13.2, formulas (13.23) and (13.24)) that the restriction of the work form $\omega^1_{\mathbf{F}}$ of a field $\mathbf{F}$ to an oriented smooth curve (path) γ or the restriction of the flux form $\omega^2_{\mathbf{V}}$ of a field $\mathbf{V}$ to an oriented surface S can be written respectively in the following forms:

$$\omega^1_{\mathbf{F}}\big|_\gamma = \langle \mathbf{F}, \mathbf{e}\rangle\, \mathrm{d}s\,, \qquad \omega^2_{\mathbf{V}}\big|_S = \langle \mathbf{V}, \mathbf{n}\rangle\, \mathrm{d}\sigma\,,$$

where $\mathbf{e}$ is the unit vector that orients γ, codirectional with the velocity vector of the motion along γ, $\mathrm{d}s$ is the element (form) of arc length on γ, $\mathbf{n}$ is the unit normal vector to S that orients the surface, and $\mathrm{d}\sigma$ is the element (form) of area on S.

In vector analysis we often use the vector element of length of a curve $\mathrm{d}\mathbf{s} := \mathbf{e}\,\mathrm{d}s$ and the vector element of area on a surface $\mathrm{d}\boldsymbol{\sigma} := \mathbf{n}\,\mathrm{d}\sigma$. Using this notation, we can now write:

$$\omega^1_{\mathbf{A}}\big|_\gamma = \langle \mathbf{A}, \mathbf{e}\rangle\, \mathrm{d}s = \langle \mathbf{A}, \mathrm{d}\mathbf{s}\rangle = \mathbf{A}\cdot \mathrm{d}\mathbf{s}\,, \tag{14.38}$$

$$\omega^2_{\mathbf{B}}\big|_S = \langle \mathbf{B}, \mathbf{n}\rangle\, \mathrm{d}\sigma = \langle \mathbf{B}, \mathrm{d}\boldsymbol{\sigma}\rangle = \mathbf{B}\cdot \mathrm{d}\boldsymbol{\sigma}\,. \tag{14.39}$$

b. The Newton–Leibniz Formula Let $f \in C^{(1)}(D,\mathbb{R})$, and let $\gamma : [a,b] \to D$ be a path in the domain D.

Applied to the 0-form ω^0_f, Stokes' formula

$$\int_{\partial\gamma} \omega^0_f = \int_\gamma \mathrm{d}\omega^0_f$$

means, on the one hand, the equality

$$\int_{\partial\gamma} f = \int_\gamma \mathrm{d}f\,,$$

which agrees with the classical formula

$$f\big(\gamma(b)\big) - f\big(\gamma(a)\big) = \int_a^b \mathrm{d}f\big(\gamma(t)\big)$$

of Newton–Leibniz (the fundamental theorem of calculus). On the other hand, by definition of the gradient, it means that

$$\int_{\partial\gamma} \omega^0_f = \int_\gamma \omega^1_{\operatorname{grad} f}\,. \tag{14.40}$$

Thus, using relation (14.38), we can rewrite the Newton–Leibniz formula as

$$\boxed{f\big(\gamma(b)\big) - f\big(\gamma(a)\big) = \int_\gamma (\operatorname{grad} f) \cdot \mathrm{d}s\ .} \tag{14.40'}$$

In this form it means that

the increment of a function on a path equals the work done by the gradient of the function on the path.

This is a very convenient and informative notation. In addition to the obvious deduction that the work of the field grad f along a path γ depends only on the endpoints of the path, the formula enables us to make a somewhat more subtle observation. To be specific, motion over a level surface $f = c$ of f takes place without any work being done by the field grad f since in this case grad $f \cdot \mathrm{d}\boldsymbol{\sigma} = 0$. Then, as the left-hand side of the formula shows, the work of the field grad f depends not even on the initial and final points of the path but only on the level surfaces of f to which they belong.

c. Stokes' Formula We recall that the work of a field on a closed path is called the *circulation of the field on that path.* To indicate that the integral is taken over a closed path, we often write $\oint_\gamma \mathbf{F} \cdot \mathrm{ds}$ rather than the traditional notation $\int_\gamma \mathbf{F} \cdot \mathrm{ds}$. If γ is a curve in the plane, we often use the symbols $\circlearrowleft\!\!\!\!\int_\gamma$ and $\circlearrowright\!\!\!\!\int_\gamma$, in which the direction of traversal of the curve γ is indicated.

The term *circulation* is also used when speaking of the integral over some finite set of closed curves. For example, it might be the integral over the boundary of a compact surface with boundary.

Let $\mathbf{A}$ be a smooth vector field in a domain D of the oriented Euclidean space $\mathbb{R}^3$ and S a (piecewise) smooth oriented compact surface with boundary in D. Applied to the 1-form $\omega^1_{\mathbf{A}}$, taking account of the definition of the curl of a vector field, Stokes' formula means the equality

$$\int_{\partial S} \omega^1_{\mathbf{A}} = \int_S \omega^2_{\operatorname{curl} \mathbf{A}}\ . \tag{14.41}$$

Using relation (14.39), we can rewrite (14.41) as the classical Stokes formula

$$\boxed{\oint_{\partial S} \mathbf{A} \cdot \mathrm{ds} = \iint_S (\operatorname{curl} \mathbf{A}) \cdot \mathrm{d}\boldsymbol{\sigma}\ .} \tag{14.41'}$$

In this notation it means that

> *the circulation of a vector field on the boundary of a surface equals the flux of the curl of the field across the surface.*

As always, the orientation chosen on ∂S is the one induced by the orientation of S.

d. The Gauss–Ostrogradskii Formula Let V be a compact domain of the oriented Euclidean space $\mathbb{R}^3$ bounded by a (piecewise-) smooth surface ∂V, the boundary of V. If $\mathbf{B}$ is a smooth field in V, then in accordance with the definition of the divergence of a field, Stokes' formula yields the equality

$$\int_{\partial V} \omega^2_{\mathbf{B}} = \int_{V} \omega^3_{\operatorname{div}\mathbf{B}} \, . \tag{14.42}$$

Using relation (14.39) and the notation $\rho\,\mathrm{d}V$ for the form ω^3_ρ in terms of the volume element $\mathrm{d}V$ in $\mathbb{R}^3$, we can rewrite Eq. (14.42) as the classical Gauss–Ostrogradskii formula

$$\boxed{\iint_{\partial V} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\sigma} = \iiint_{V} \operatorname{div}\mathbf{B}\,\mathrm{d}V\,.} \tag{14.42$'$}$$

In this form it means that

> *the flux of a vector field across the boundary of a domain equals the integral of the divergence of the field over the domain itself.*

e. Summary of the Classical Integral Formulas In sum, we have arrived at the following vector notation for the three classical integral formulas of analysis:

$$\int_{\partial\gamma} f = \int_{\gamma} (\nabla f)\cdot \mathrm{ds} \ \text{ (the Newton–Leibniz formula)}\,, \tag{14.40$''$}$$

$$\int_{\partial S} \mathbf{A}\cdot \mathrm{ds} = \int_{S} (\nabla\times\mathbf{A})\cdot \mathrm{d}\boldsymbol{\sigma} \quad \text{(Stokes' formula)}\,, \tag{14.41$''$}$$

$$\int_{\partial V} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\sigma} = \int_{V} (\nabla\cdot\mathbf{B})\,\mathrm{d}V \ \text{(the Gauss–Ostrogradskii formula)}\,. \tag{14.42$''$}$$

14.2.2 The Physical Interpretation of div, curl, and grad

a. The Divergence Formula (14.42$'$) can be used to explain the physical meaning of $\operatorname{div}\mathbf{B}(x)$ – the divergence of the vector field $\mathbf{B}$ at a point x in the domain V in which the field is defined. Let $V(x)$ be a neighborhood of x (for example, a ball) contained in V. We permit ourselves to denote the volume of this neighborhood by the same symbol $V(x)$ and its diameter by the letter d.

By the mean-value theorem and the formula (14.42′) we obtain the following relation for the triple integral

$$\iint\limits_{\partial V(x)} \mathbf{B} \cdot d\boldsymbol{\sigma} = \operatorname{div} \mathbf{B}(x') V(x) ,$$

where x' is a point in the neighborhood $V(x)$. If $d \to 0$, then $x' \to x$, and since $\mathbf{B}$ is a smooth field, we also have $\operatorname{div} \mathbf{B}(x') \to \operatorname{div} \mathbf{B}(x)$. Hence

$$\operatorname{div} \mathbf{B}(x) = \lim_{d \to 0} \frac{\iint\limits_{\partial V(x)} \mathbf{B} \cdot d\boldsymbol{\sigma}}{V(x)} . \tag{14.43}$$

Let us regard $\mathbf{B}$ as the velocity field for a flow (of liquid or gas). Then, by the law of conservation of mass, a flux of this field across the boundary of the domain V or, what is the same, a volume of the medium diverging across the boundary of the domain, can arise only when there are sinks or sources (including those associated with a change in the density of the medium). The flux is equal to the total power of all these factors, which we shall collectively call "sources," in the domain $V(x)$. Hence the fraction on the right-hand side of (14.43) is the mean intensity (per unit volume) of sources in the domain $V(x)$, and the limit of that quantity, that is, $\operatorname{div} \mathbf{B}(x)$, is the specific intensity (per unit volume) of the source at the point x. But the limit of the ratio of the total amount of some quantity in the domain $V(x)$ to the volume of that domain as $d \to 0$ is customarily called the *density* of that quantity at x, and the density as a function of a point is usually called the *density of the distribution* of the given quantity in a portion of space.

Thus, we can interpret the divergence $\operatorname{div} \mathbf{B}$ of a vector field B as the density of the distribution of sources in the domain of the flow, that is, in the domain of definition of the field $\mathbf{B}$.

Example 1. If, in particular, $\operatorname{div} \mathbf{B} \equiv 0$, that is, there are no sources, then the flux across the boundary of the region must be zero: the amount flowing in equals the amount flowing out. And, as formula (14.42′) shows, this is indeed the case.

Example 2. A point electric charge of magnitude q creates an electric field in space. Suppose the charge is located at the origin. By *Coulomb's law*[6] the intensity $\mathbf{E} = E(x)$ of the field at the point $x \in \mathbb{R}^3$ (that is, the force acting

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[6] Ch.O. Coulomb (1736–1806) – French physicist. He discovered experimentally the law (Coulomb's law) of interaction of charges and magnetic fields using a torsion balance that he invented himself.

on a unit test charge at the point x) can be written as

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3} ,$$

where ε_0 is a dimensioning constant and $\mathbf{r}$ is the radius-vector of the point x.

The field $\mathbf{E}$ is defined at all points different from the origin. In spherical coordinates $\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\mathbf{e}_R$, so that by formula (14.36‴) of the preceding section, one can see immediately that $\operatorname{div}\mathbf{E} = 0$ everywhere in the domain of definition of the field $\mathbf{E}$.

Hence, if we take any domain V not containing the origin, then by formula (14.42′) the flux of $\mathbf{E}$ across the boundary ∂V of V is zero.

Let us now take the sphere $S_R = \{x \in \mathbb{R}^3 \big| |x| = R\}$ of radius R with center at the origin and find the outward flux (relative to the ball bounded by the sphere) of $\mathbf{E}$ across this surface. Since the vector $\mathbf{e}_R$ is itself the unit outward normal to the sphere, we find

$$\int\limits_{S_R} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\sigma} = \int\limits_{S_R} \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\,\mathrm{d}\sigma = \frac{q}{4\pi\varepsilon_0 R^2}\cdot 4\pi R^2 = \frac{q}{\varepsilon_0} .$$

Thus, up to the dimensioning constant ε_0, which depends on the choice of the system of physical units, we have found the amount of charge in the volume bounded by the sphere.

We remark that under the hypotheses of Example 2 just studied the left-hand side of formula (14.42′) is well-defined on the sphere $\partial V = S_R$, but the integrand on the right-hand side is defined and equal to zero everywhere in the ball V except at one point – the origin. Nevertheless, the computations show that the integral on the right-hand side of (14.42′) cannot be interpreted as the integral of a function that is identically zero.

From the formal point of view one could dismiss the need to study this situation by saying that the field $\mathbf{E}$ is not defined at the point $0 \in V$, and hence we do not have the right to speak about the equality (14.42′), which was proved for smooth fields defined in the entire domain V of integration. However, the physical interpretation of (14.42′) as the law of conservation of mass shows that, when suitably interpreted, it ought to be valid always.

Let us study the indeterminacy of $\operatorname{div}\mathbf{E}$ at the origin in Example 2 more attentively to see what is causing it. Formally the original field $\mathbf{E}$ is not defined at the origin, but, if we seek $\operatorname{div}\mathbf{E}$ from formula (14.43), then, as Example 2 shows, we would have to assume that $\operatorname{div}\mathbf{E}(0) = +\infty$. Hence the integrand on the right-hand side of (14.42) would be a "function" equal to zero everywhere except at one point, where it is equal to infinity. This corresponds to the fact that there are no charges at all outside the origin, and we somehow managed to put the entire charge q into a space of volume zero – into the single point 0, at which the charge density naturally became infinite. Here we are encountering the so-called Dirac[7] δ-function (delta-function).

The densities of physical quantities are needed ultimately so that one can find the values of the quantities themselves by integrating the density. For that reason there is no need to define the δ-function at each individual point; it is more important to define its integral. If we assume that physically the "function" $\delta_{x_0}(x) = \delta(x_0; x)$ must correspond to the density of a distribution, for example the distribution of mass in space, for which the entire mass, equal to 1 in magnitude, is concentrated at the single point x_0, it is natural to set

$$\int\limits_V \delta(x_0, x)\, \mathrm{d}V = \begin{cases} 1\,, & \text{when } x_0 \in V\,, \\ 0\,, & \text{when } x_0 \notin V\,. \end{cases}$$

Thus, from the point of view of a mathematical idealization of our ideas of the possible distribution of a physical quantity (mass, charge, and the like) in space, we must assume that its distribution density is the sum of an ordinary finite function corresponding to a continuous distribution of the quantity in space and a certain set of singular "functions" (of the same type as the Dirac δ-function) corresponding to a concentration of the quantity at individual points of space.

Hence, starting from these positions, the results of the computations in Example 2 can be expressed as the single equality $\operatorname{div} \mathbf{E} = \frac{q}{\varepsilon_0}\delta(0; x)$. Then, as applied to the field $\mathbf{E}$, the integral on the right-hand side of (14.42′) is indeed equal either to q/ε_0 or to 0, according as the domain V contains the origin (and the point charge concentrated there) or not.

In this sense one can assert (following Gauss) that the flux of electric field intensity across the surface of a body equals (up to a factor depending on the units chosen) the sum of the electric charges contained in the body. In this same sense one must interpret the electric charge density ρ in the Maxwell equations considered in Sect. 14.1 (formula (14.12)).

b. The Curl We begin our study of the physical meaning of the curl with an example.

Example 3. Suppose the entire space, regarded as a rigid body, is rotating with constant angular speed ω about a fixed axis (let it be the x-axis). Let us find the curl of the field $\mathbf{v}$ of linear velocities of the points of space. (The field is being studied at any fixed instant of time.)

In cylindrical coordinates (r, φ, z) we have the simple expression $\mathbf{v}(r, \varphi, z) = \omega r \mathbf{e}_\varphi$. Then by formula (14.35″) of Sect. 14.1, we find immediatly that $\operatorname{curl} \mathbf{v} = 2\omega \mathbf{e}_z$. That is, $\operatorname{curl} \mathbf{v}$ is a vector directed along the axis

[7] P.A.M. Dirac (1902–1984) – British theoretical physicist, one of the founders of quantum mechanics. More details on the Dirac δ-function will be given in Subsects. 17.4.4 and 17.5.4.

of rotation. Its magnitude 2ω equals the angular velocity of the rotation, up to the coefficient 2, and the direction of the vector, taking account of the orientation of the whole space $\mathbb{R}^3$, completely determines the direction of rotation.

The field described in Example 3 in the small resembles the velocity field of a funnel (sink) or the field of the vorticial motion of air in the neighborhood of a tornado (also a sink, but one that drains upward). Thus, the curl of a vector field at a point characterizes the degree of vorticity of the field in a neighborhood of the point.

We remark that the circulation of a field over a closed contour varies in direct proportion to the magnitude of the vectors in the field, and, as one can verify using the same Example 3, it can also be used to characterize the vorticity of the field. Only now, to describe completely the vorticity of the field in a neighborhood of a point, it is necessary to compute the circulation over contours lying in three different planes. Let us now carry out this program.

We take a disk $S_i(x)$ with center at the point x and lying in a plane perpendicular to the ith coordinate axis, $i = 1, 2, 3$. We orient $S_i(x)$ using a normal, which we take to be the unit vector $\mathbf{e}_i$ along this coordinate axis. Let d be the diameter of $S_i(x)$. From formula (14.41) for a smooth field $\mathbf{A}$ we find that

$$(\operatorname{curl} \mathbf{A}) \cdot \mathbf{e}_i = \lim_{d \to 0} \frac{\oint\limits_{\partial S_i(x)} \mathbf{A} \cdot d\mathbf{s}}{S_i(x)} , \tag{14.44}$$

where $S_i(x)$ denotes the area of the disk under discussion. Thus the circulation of the field $\mathbf{A}$ over the boundary ∂S_i per unit area in the plane orthogonal to the ith coordinate axis characterizes the ith component of $\operatorname{curl} \mathbf{A}$.

To clarify still further the meaning of the curl of a vector field, we recall that every linear transformation of space is a composition of dilations in three mutually perpendicular directions, translation of the space as a rigid body, and rotation as a rigid body. Moreover, every rotation can be realized as a rotation about some axis. Every smooth deformation of the medium (flow of a liquid or gas, sliding of the ground, bending of a steel rod) is locally linear. Taking account of what has just been said and Example 3, we can conclude that if there is a vector field that describes the motion of a medium (the velocity field of the points in the medium), then the curl of that field at each point gives the instantaneous axis of rotation of a neighborhood of the point, the magnitude of the instantaneous angular velocity, and the direction of rotation about the instantaneous axis. That is, the curl characterizes completely the rotational part of the motion of the medium. This will be made slightly more precise below, where it will be shown that the curl should be regarded as a sort of density for the distribution of local rotations of the medium.

c. The Gradient We have already said quite a bit about the gradient of a scalar field, that is, about the gradient of a function. Hence at this point we shall merely recall the main things.

Since $\omega^1_{\text{grad}\, f}(\boldsymbol{\xi}) = \langle \text{grad}\, f, \boldsymbol{\xi}\rangle = \mathrm{d}f(\boldsymbol{\xi}) = D_{\boldsymbol{\xi}} f$, where $D_{\boldsymbol{\xi}} f$ is the derivative of the function f with respect to the vector $\boldsymbol{\xi}$, it follows that grad f is orthogonal to the level surfaces of f, and at each point it points in the direction of most rapid increase in the values of the function. Its magnitude $|\text{grad}\, f|$ gives the rate of that growth (per unit of length in the space in which the argument varies).

The significance of the gradient as a density will be discussed below.

14.2.3 Other Integral Formulas

a. Vector Versions of the Gauss–Ostrogradskii Formula The interpretation of the curl and gradient as vector densities, analogous to the interpretation (14.43) of the divergence as a density, can be obtained from the following classical formulas of vector analysis, connected with the Gauss–Ostrogradskii formula.

$$\int_V \nabla \cdot \mathbf{B}\, \mathrm{d}V = \int_{\partial V} \mathrm{d}\boldsymbol{\sigma} \cdot B \quad \text{(the divergence theorem)} \tag{14.45}$$

$$\int_V \nabla \times \mathbf{A}\, \mathrm{d}V = \int_{\partial V} \mathrm{d}\boldsymbol{\sigma} \times \mathbf{A} \quad \text{(the curl theorem)} \tag{14.46}$$

$$\int_V \nabla f\, \mathrm{d}V = \int_{\partial V} \mathrm{d}\boldsymbol{\sigma} f \quad \text{(the gradient theorem)} \tag{14.47}$$

The first of these three relations coincides with (14.42′) up to notation and is the Gauss–Ostrogradskii formula. The vector equalities (14.46) and (14.47) follow from (14.45) if we apply that formula to each component of the corresponding vector field.

Retaining the notation $V(x)$ and d used in Eq. (14.43), we obtain from formulas (14.45)–(14.47) in a unified manner,

$$\nabla \cdot \mathbf{B}(x) = \lim_{d\to 0} \frac{\int\limits_{\partial V(x)} \mathrm{d}\boldsymbol{\sigma} \cdot \mathbf{B}}{V(x)}\,, \tag{14.43′}$$

$$\nabla \times \mathbf{A}(x) = \lim_{d\to 0} \frac{\int\limits_{\partial V(x)} \mathrm{d}\boldsymbol{\sigma} \times \mathbf{A}}{V(x)} \tag{14.48}$$

$$\nabla f(x) = \lim_{d\to 0} \frac{\int\limits_{\partial V(x)} \mathrm{d}\boldsymbol{\sigma} f}{V(x)}\,. \tag{14.49}$$

The right-hand sides of (14.45)–(14.47) can be interpreted respectively as the scalar flux of the vector field $\mathbf{B}$, the vector flux of the vector field $\mathbf{A}$, and the vector flux of the scalar field f across the surface ∂V bounding the domain V. Then the quantities $\operatorname{div}\mathbf{B}$, $\operatorname{curl}\mathbf{A}$, and $\operatorname{grad} f$ on the left-hand sides of Eqs. (14.43′), (14.48), and (14.49) can be interpreted as the corresponding source densities of these fields.

We remark that the right-hand sides of Eqs. (14.43′), (14.48), and (14.49) are independent of the coordinate system. From these we can once again derive the invariance of the gradient, curl, and divergence.

b. Vector Versions of Stokes' Formula Just as formulas (14.45)–(14.47) were the result of combining the Gauss–Ostrogradskii formula with the algebraic operations on vector and scalar fields, the following triple of formulas can be obtained by combining these same operations with the classical Stokes formula (which appears as the first of the three relations).

Let S be a (piecewise-) smooth compact oriented surface with a consistently oriented boundary ∂S, let $\mathrm{d}\boldsymbol{\sigma}$ be the vector element of area on S, and ds the vector element of length on ∂S. Then for smooth fields $\mathbf{A}$, $\mathbf{B}$, and f, the following relations hold:

$$\int_S \mathrm{d}\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{A}) = \int_{\partial S}\mathrm{ds}\cdot\mathbf{A}\,, \tag{14.50}$$

$$\int_S (\mathrm{d}\boldsymbol{\sigma}\times\nabla)\times\mathbf{B} = \int_{\partial S}\mathrm{ds}\times\mathbf{B}\,, \tag{14.51}$$

$$\int_S \mathrm{d}\boldsymbol{\sigma}\times\nabla f = \int_{\partial S}\mathrm{ds}f\,. \tag{14.52}$$

Formulas (14.51) and (14.52) follow from Stokes' formula (14.50). We shall not take time to give the proofs.

c. Green's Formulas If S is a surface and $\mathbf{n}$ a unit normal vector to S, then the derivative $D_{\mathbf{n}}f$ of the function f with respect to $\mathbf{n}$ is usually denoted $\frac{\partial f}{\partial n}$ in field theory. For example, $\langle\nabla f,\mathrm{d}\boldsymbol{\sigma}\rangle = \langle\nabla f,\mathbf{n}\rangle\,\mathrm{d}\sigma = \langle\operatorname{grad} f,\mathbf{n}\rangle\,\mathrm{d}\sigma = D_{\mathbf{n}}f\,\mathrm{d}\sigma = \frac{\partial f}{\partial n}\,\mathrm{d}\sigma$. Thus, $\frac{\partial f}{\partial n}\,\mathrm{d}\sigma$ is the flux of $\operatorname{grad} f$ across the element of surface $\mathrm{d}\sigma$.

In this notation we can write the following formulas of Green, which are very widely used in analysis:

$$\int_V \nabla f\cdot\nabla g\,\mathrm{d}V + \int_V g\nabla^2 f\,\mathrm{d}V = \int_{\partial V}(g\nabla f)\cdot\mathrm{d}\boldsymbol{\sigma}\quad\left(=\int_{\partial V} g\frac{\partial f}{\partial n}\,\mathrm{d}\sigma\right), \tag{14.53}$$

$$\int_V (g\nabla^2 f - f\nabla^2 g)\,\mathrm{d}V = \int_{\partial V}(g\nabla f - f\nabla g)\cdot\mathrm{d}\boldsymbol{\sigma}\left(=\int_{\partial V}\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\mathrm{d}\sigma\right). \tag{14.54}$$

In particular, if we set $f = g$ in (14.53) and $g \equiv 1$ in (14.54), we find respectively,

$$\int\limits_V |\nabla f|^2 \,\mathrm{d}V + \int\limits_V f\Delta f \,\mathrm{d}V = \int\limits_{\partial V} f\nabla f \cdot \mathrm{d}\boldsymbol{\sigma}\left(= \int\limits_{\partial V} f \frac{\partial f}{\partial n} \,\mathrm{d}\sigma\right), \quad (14.53')$$

$$\int\limits_V \Delta f \,\mathrm{d}V = \int\limits_{\partial V} \nabla f \cdot \mathrm{d}\boldsymbol{\sigma}\left(= \int\limits_{\partial V} \frac{\partial f}{\partial n} \,\mathrm{d}\sigma\right). \quad (14.54')$$

This last equality is often called *Gauss' theorem*. Let us prove, for example, the second of Eqs. (14.53) and (14.54):

Proof.

$$\int\limits_{\partial V} (g\nabla f - f\nabla g) \cdot \mathrm{d}\boldsymbol{\sigma} = \int\limits_V \nabla \cdot (g\nabla f - f\nabla g) \,\mathrm{d}V =$$

$$= \int\limits_V (\nabla g \cdot \nabla f + g\nabla^2 f - \nabla f \cdot \nabla g - f\nabla^2 g) \,\mathrm{d}V =$$

$$= \int\limits_V (g\nabla^2 f - f\nabla^2 g) \,\mathrm{d}V = \int\limits_V (g\Delta f - f\Delta g) \,\mathrm{d}V .$$

In this formula we have used the Gauss–Ostrogradskii formula and the relation $\nabla \cdot (\varphi \mathbf{A}) = \nabla\varphi \cdot \mathbf{A} + \varphi\nabla \cdot \mathbf{A}$. □

14.2.4 Problems and Exercises

1. Using the Gauss–Ostrogradskii formula (14.45), prove relations (14.46) and (14.47).

2. Using Stokes' formula (14.50), prove relations (14.51) and (14.52).

3. a) Verify that formulas (14.45), (14.46), and (14.47) remaind valid for an unbounded domain V if the integrands in the surface integrals are of order $O\left(\frac{1}{r^3}\right)$ as $r \to \infty$. (Here $r = |\mathbf{r}|$, and $\mathbf{r}$ is the radius-vector in $\mathbb{R}^3$.)

b) Determine whether formulas (14.50), (14.51), and (14.52) remain valid for a noncompact surface $S \subset \mathbb{R}^3$ if the integrands in the line integrals are of order $O\left(\frac{1}{r^2}\right)$ as $r \to \infty$.

c) Give examples showing that for unbounded surfaces and domains Stokes' formula (14.41′) and the Gauss–Ostrogradskii formula (14.42′) are in general not true.

4. a) Starting from the interpretation of the divergence as a source density, explain why the second of the Maxwell equations (formula (14.12) of Sect. 14.1) implies that there are no point sources in the magnetic field (that is, there are no magnetic charges).

b) Using the Gauss–Ostrogradskii formula and the Maxwell equations (formula (14.12) of Sect. 14.1), show that no rigid configuration of test charges (for example a single charge) can be in a stable equilibrium state in the domain of an electrostatic field that is free of the (other) charges that create the field. (It is assumed that no forces except those exerted by the field act on the system.) This fact is known as *Earnshaw's theorem*.

5. If an electromagnetic field is steady, that is, independent of time, then the system of Maxwell equations (formula (14.12) of Sect. 14.1) decomposes into two independent parts – the *electrostatic equations* $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$, $\nabla \times \mathbf{E} = \mathbf{0}$, and the *magnetostatic equations* $\nabla \times \mathbf{B} = \frac{\mathbf{j}}{\varepsilon_0 c^2}$, $\nabla \cdot \mathbf{B} = 0$.

The equation $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$, where ρ is the charge density, transforms via the Gauss–Ostrogradskii formula into $\int\limits_S \mathbf{E} \cdot \mathrm{d}\boldsymbol{\sigma} = Q/\varepsilon_0$, where the left-hand side is the flux of the electric field intensity across the closed surface S and the right-hand side is the sum Q of the charges in the domain bounded by S, divided by the dimensioning constant ε_0. In electrostatics this relation is usually called *Gauss' law*. Using Gauss' law, find the electric field $\mathbf{E}$

a) created by a uniformly charged sphere, and verify that outside the sphere it is the same as the field of a point charge of the same magnitude located at the center of the sphere;

b) of a uniformly charged line;

c) of a uniformly charged plane;

d) of a pair of parallel planes uniformly charged with charges of opposite sign;

e) of a uniformly charged ball.

6. a) Prove Green's formula (14.53).

b) Let f be a *harmonic function* in the bounded domain V (that is, f satisfies Laplace's equation $\Delta f = 0$ in V). Show, starting from (14.54′) that the flux of the gradient of this function across the boundary of the domain V is zero.

c) Verify that a harmonic function in a bounded connected domain is determined up to an additive constant by the values of its normal derivative on the boundary of the domain.

d) Starting from (14.53′), prove that if a harmonic function in a bounded domain vanishes on the boundary, it is identically zero throughout the domain.

e) Show that if the values of two harmonic functions are the same on the boundary of a bounded domain, then the functions are equal in the domain.

f) Starting from (14.53), verify the following *principle of Dirichlet. Among all continuous differentiable functions in a domain assuming prescribed values on the boundary, a harmonic function in the region is the only one that minimizes the Dirichlet integral (that is, the integral of the squared-modulus of the gradient over the domain).*

7. a) Let $r(p,q) = |p-q|$ be the distance between the points p and q in the Euclidean space $\mathbb{R}^3$. By fixing p, we obtain a function $r_p(q)$ of $q \in \mathbb{R}^3$. Show that $\Delta r_p^{-1}(q) = 4\pi\delta(p;q)$, where δ is the δ-function.

b) Let g be harmonic in the domain V. Setting $f = 1/r_p$ in (14.54) and taking account of the preceding result, we obtain

$$4\pi g(p) = \int\limits_S \left(g\nabla \frac{1}{r_p} - \frac{1}{r_p}\nabla g \right) \cdot \mathrm{d}\boldsymbol{\sigma} .$$

Prove this equality precisely.

c) Deduce from the preceding equality that if S is a sphere of radius R with center at p, then

$$g(p) = \frac{1}{4\pi R^2} \int\limits_S g \, \mathrm{d}\sigma .$$

This is the so-called *mean-value theorem* for harmonic functions.

d) Starting from the preceding result, show that if B is the ball bounded by the sphere S considered in part c) and $V(B)$ is its volume, then

$$g(p) = \frac{1}{V(B)} \int\limits_B g \, \mathrm{d}V .$$

e) If p and q are points of the Euclidean plane $\mathbb{R}^2$, then along with the function $\frac{1}{r_p}$ considered in a) above (corresponding to the potential of a charge located at p), we now take the function $\ln \frac{1}{r_p}$ (corresponding to the potential of a uniformly charged line in space). Show that $\Delta \ln \frac{1}{r_p} = 2\pi\delta(p;q)$, where $\delta(p;q)$ is now the δ-function in $\mathbb{R}^2$.

f) By repeating the reasoning in a), b), c), and d), obtain the mean-value theorem for functions that are harmonic in plane regions.

8. *Cauchy's multi-dimensional mean-value theorem.*

The classical mean-value theorem for the integral ("Lagrange's theorem") asserts that if the function $f : D \to \mathbb{R}$ is continuous on a compact, measurable, and connected set $D \subset \mathbb{R}^n$ (for example, in a domain), then there exists a point $\xi \in D$ such that

$$\int\limits_D f(x)\, \mathrm{d}x = f(\xi) \cdot |D| ,$$

where $|D|$ is the measure (volume) of D.

a) Now let $f, g \in C(D, \mathbb{R})$, that is, f and g are continuous real-valued functions in D. Show that the following theorem ("Cauchy's theorem") holds: *There exists* $\xi \in D$ *such that*

$$g(\xi) \int\limits_D f(x)\, \mathrm{d}x = f(\xi) \int\limits_D g(x)\, \mathrm{d}x .$$

b) Let D be a compact domain with smooth boundary ∂D and $\mathbf{f}$ and $\mathbf{g}$ two smooth vector fields in D. Show that there exists a point $\xi \in D$ such that

$$\operatorname{div} \mathbf{g}(\xi) \cdot \operatorname*{Flux}_{\partial D} \mathbf{f} = \operatorname{div} \mathbf{f}(\xi) \cdot \operatorname*{Flux}_{\partial D} \mathbf{g} ,$$

where $\operatorname*{Flux}_{\partial D}$ is the flux of a vector field across the surface ∂D.

14.3 Potential Fields

14.3.1 The Potential of a Vector Field

Definition 1. Let $\mathbf{A}$ be a vector field in the domain $D \subset \mathbb{R}^n$. A function $U : D \to \mathbb{R}$ is called a *potential of the field* $\mathbf{A}$ if $\mathbf{A} = \operatorname{grad} U$ in D.

Definition 2. A field that has a potential is called a *potential field.*

Since the partial derivatives of a function determine the function up to an additive constant in a connected domain, the potential is unique in such a domain up to an additive constant.

We briefly mentioned potentials in the first part of this course. Now we shall discuss this important concept in somewhat more detail. In connection with these definitions we note that when different force fields are studied in physics, the potential of a field $\mathbf{F}$ is usually defined as a function U such that $\mathbf{F} = -\operatorname{grad} U$. This potential differs from the one given in Definition 1 only in sign.

Example 1. At a point of space having radius-vector $\mathbf{r}$ the intensity $\mathbf{F}$ of the gravitational field due to a point mass M located at the origin can be computed from Newton's law as

$$\mathbf{F} = -GM\frac{\mathbf{r}}{r^3}\,, \tag{14.55}$$

where $r = |\mathbf{r}|$.

This is the force with which the field acts on a unit mass at this point of space. The gravitational field (14.55) is a potential field. Its potential in the sense of Definition 1 is the function

$$U = GM\frac{1}{r}\,. \tag{14.56}$$

Example 2. At a point of space having radius-vector $\mathbf{r}$ the intensity $\mathbf{E}$ of the electric field due to a point charge q located at the origin can be computed from Coulomb's law

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\,.$$

Thus such an electrostatic field, like the gravitational field, is a potential field. Its potential φ in the sense of physical terminology is defined by the relation

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\,.$$

14.3.2 Necessary Condition for Existence of a Potential

In the language of differential forms the equality $\mathbf{A} = \operatorname{grad} U$ means that $\omega^1_{\mathbf{A}} = \mathrm{d}\omega^0_U = \mathrm{d}U$, from which it follows that

$$\mathrm{d}\omega^1_{\mathbf{A}} = 0 , \tag{14.57}$$

since $\mathrm{d}^2\omega^0_U = 0$. This is a necessary condition for the field $\mathbf{A}$ to be a potential field.

In Cartesian coordinates this condition can be expressed very simply. If $\mathbf{A} = (A^1, \dots, A^n)$ and $\mathbf{A} = \operatorname{grad} U$, then $A^i = \frac{\partial U}{\partial x^i}$, $i = 1, \dots, n$, and if the potential U is sufficiently smooth (for example, if its second-order partial derivatives are continuous), we must have

$$\frac{\partial A^i}{\partial x^j} = \frac{\partial A^j}{\partial x^i} , \qquad i, j = 1, \dots, n , \tag{14.57'}$$

which simply means that the mixed partial derivatives are equal in both orders:

$$\frac{\partial^2 U}{\partial x^i \partial x^j} = \frac{\partial^2 U}{\partial x^j \partial x^i} .$$

In Cartesian coordinates $\omega^1_{\mathbf{A}} = \sum\limits_{i=1}^{n} A^i \, \mathrm{d}x^i$, and therefore the equalities (14.57) and (14.57′) are indeed equivalent in this case.

In the case of $\mathbb{R}^3$ we have $\mathrm{d}\omega^1_{\mathbf{A}} = \omega^2_{\operatorname{curl} \mathbf{A}}$, so that the necessary condition (14.57) can be rewritten as

$$\operatorname{curl} \mathbf{A} = 0 ,$$

which corresponds to the relation $\operatorname{curl} \operatorname{grad} U = \mathbf{0}$, which we already know.

Example 3. The field $\mathbf{A} = (x, xy, xyz)$ in Cartesian coordinates in $\mathbb{R}^3$ cannot be a potential field, since, for example, $\frac{\partial xy}{\partial x} \neq \frac{\partial x}{\partial y}$.

Example 4. Consider the field $\mathbf{A} = (A_x, A_y)$ given by

$$\mathbf{A} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) , \tag{14.58}$$

defined in Cartesian coordinates at all points of the plane except the origin. The necessary condition for a field to be a potential field $\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x}$ is fulfilled in this case. However, as we shall soon verify, this field is not a potential field in its domain of definition.

Thus the necessary condition (14.57), or, in Cartesian coordinates (14.57′), is in general not sufficient for a field to be a potential field.

14.3.3 Criterion for a Field to be Potential

Proposition 1. *A continuous vector field* **A** *in a domain* $D \subset \mathbb{R}^n$ *is a potential field in* D *if and only if its circulation (work) around every closed curve* γ *contained in* D *is zero:*

$$\oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{s} = 0 . \tag{14.59}$$

Proof. Necessity. Suppose $\mathbf{A} = \operatorname{grad} U$. Then by the Newton–Leibniz formula (Formula (14.40′) of Sect. 14.2),

$$\oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{s} = U\big(\gamma(b)\big) - U\big(\gamma(a)\big) ,$$

where $\gamma : [a, b] \to D$. If $\gamma(a) = \gamma(b)$, that is, when the path γ is closed, it is obvious that the right-hand side of this last equality vanishes, and hence the left-hand side does also.

Sufficiency. Suppose condition (5) holds. Then the integral over any (not necessarily closed) path in D depends only on its initial and terminal points, not on the path joining them. Indeed, if γ_1 and γ_2 are two paths having the same initial and terminal points, then, traversing first γ_1, then $-\gamma_2$ (that is, traversing γ_2 in the opposite direction), we obtain a closed path γ whose integral, by (14.59), equals zero, but is also the difference of the integrals over γ_1 and γ_2. Hence these last two integrals really are equal.

We now fix some point $x_0 \in D$ and set

$$U(x) = \int_{x_0}^{x} \mathbf{A} \cdot \mathrm{d}\mathbf{s} , \tag{14.60}$$

where the integral on the right is the integral over any path in D from x_0 to x. We shall verify that the function U so defined is the required potential for the field **A**. For convenience, we shall assume that a Cartesian coordinate system $(x^1, \dots, x^n)$ has been chosen in $\mathbb{R}^n$. Then $\mathbf{A} \cdot \mathrm{d}\mathbf{s} = A^1\,\mathrm{d}x^1 + \dots + A^n\,\mathrm{d}x^n$. If we move away from x along a straight line in the direction $h\mathbf{e}_i$, where $\mathbf{e}_i$ is the unit vector along the x^i-axis, the function U receives an increment equal to

$$U(x + h\mathbf{e}_i) - U(x) = \int_{x^i}^{x^i+h^i} A^i(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\,\mathrm{d}t ,$$

equal to the integral of the form $\mathbf{A} \cdot \mathrm{d}\mathbf{s}$ over this path from x to $x + h\mathbf{e}_i$. By the continuity of **A** and the mean-value theorem, this last equality can be written as

$$U(x + h\mathbf{e}_i) - U(x) = A^i(x^1, \dots, x^{i-1}, x^i + \theta h, x^{i+1}, \dots, x^n)h ,$$

where $0 \le \theta \le 1$. Dividing this last equality by h and letting h tend to zero, we find

$$\frac{\partial U}{\partial x^i}(x) = A^i(x) ,$$

that is, $\mathbf{A} = \operatorname{grad} U$. □

Remark 1. As can be seen from the proof, a sufficient condition for a field to be a potential field is that (14.59) hold for smooth paths or, for example, for broken lines whose links are parallel to the coordinate axes.

We now return to Example 4. Earlier (Example 1 of Sect. 8.1) we computed that the circulation of the field (14.58) over the circle $x^2 + y^2 = 1$ traversed once in the counterclockwise direction was 2π ($\neq 0$).

Thus, by Proposition 1 we can conclude that the field (14.58) is not a potential field in the domain $\mathbb{R}^2 \setminus 0$.

But surely, for example,

$$\operatorname{grad} \arctan \frac{y}{x} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) ,$$

and it would seem that the function $\arctan \frac{y}{x}$ is a potential for (14.58). What is this, a contradiction?! There is no contradiction as yet, since the only correct conclusion that one can make in this situation is that the function $\arctan \frac{y}{x}$ is not defined in the entire domain $\mathbb{R}^2 \setminus 0$. And that is indeed the case: Take for example, the points on the y-axis. But then, you may say, we could consider the function $\varphi(x, y)$, the polar angular coordinate of the point (x, y). That is practically the same thing as $\arctan \frac{y}{x}$, but $\varphi(x, y)$ is also defined for $x = 0$, provided the point (x, y) is not at the origin. Throughout the domain $\mathbb{R}^2 \setminus 0$ we have

$$\mathrm{d}\varphi = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y .$$

However, there is still no contradiction, although the situation is now more delicate. Please note that in fact φ is not a continuous single-valued function of a point in the domain $\mathbb{R}^2 \setminus 0$. As a point encircles the origin counterclockwise, its polar angle, varying continuously, will have increased by 2π when the point returns to its starting position. That is, we arrive at the original point with a new value of the function, different from the one we began with. Consequently, we must give up either the continuity or the single-valuedness of the function φ in the domain $\mathbb{R}^2 \setminus 0$.

In a small neighborhood (not containing the origin) of each point of the domain $\mathbb{R}^2 \setminus 0$ one can distinguish a continuous single-valued branch of the function φ. All such branches differ from one another by an additive constant, a multiple of 2π. That is why they all have the same differential and can all serve locally as potentials of the field (14.58). Nevertheless, the field (14.58) has no potential in the entire domain $\mathbb{R}^2 \setminus 0$.

The situation studied in Example 4 turns out to be typical in the sense that the necessary condition (14.57) or (14.57′) for the field **A** to be a potential field is locally also sufficient. The following proposition holds.

Proposition 2. *If the necessary condition for a field to be a potential field holds in a ball, then the field has a potential in that ball.*

Proof. For the sake of intuitiveness we first carry out the proof in the case of a disk $D = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < r\}$ in the plane $\mathbb{R}^2$. One can arrive at the point (x,y) of the disk from the origin along two different two-link broken lines γ_1 and γ_2 with links parallel to the coordinate axes (see Fig. 14.3). Since D is a convex domain, the entire rectangle I bounded by these lines is contained in D.

By Stokes' formula, taking account of condition (14.57), we obtain

$$\int_{\partial I} \omega_{\mathbf{A}}^1 = \int_I \mathrm{d}\omega_{\mathbf{A}}^1 = 0\,.$$

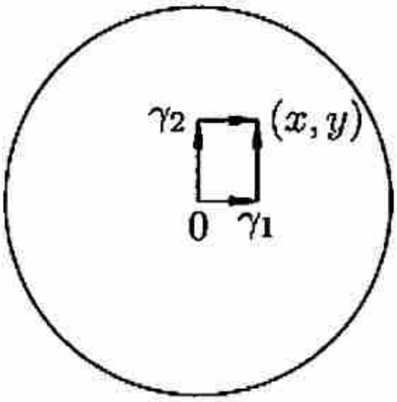

Fig. 14.3.

By the remark to Proposition 1 we can conclude from this that the field **A** is a potential field in D. Moreover, by the proof of sufficiency in Proposition 1, the function (14.60) can again be taken as the potential, the integral being interpreted as the integral over a broken line from the center to the point in question with links parallel to the axes. In this case the independence of the choice of path γ_1, γ_2 for such an integral followed immediately from Stokes' formula for a rectangle.

In higher dimensions it follows from Stokes' formula for a two-dimensional rectangle that replacing two adjacent links of the broken line by two links forming the sides of a rectangle parallel to the original does not change the value of the integral over the path. Since one can pass from one broken-line path to any other broken-line path leading to the same point by a sequence of such reconstructions, the potential is unambiguously defined in the general case. □

14.3.4 Topological Structure of a Domain and Potentials

Comparing Example 4 and Proposition 2, one can conclude that when the necessary condition (14.57) for a field to be a potential field holds, the question whether it is always a potential field depends on the (topological) structure of the domain in which the field is defined. The following considerations (here and in Subsect. 14.3.5 below) give an elementary idea as to exactly how the characteristics of the domain bring this about.

It turns out that if the domain D is such that every closed path in D can be contracted to a point of the domain without going outside the domain, then the necessary condition (14.57) for a field to be a potential field in D is also sufficient. We shall call such domains *simply connected* below. A ball is a simply connected domain (and that is why Proposition 2 holds). But the punctured plane $\mathbb{R}^2 \setminus 0$ is not simply connected, since a path that encircles the origin cannot be contracted to a point without going outside the region. This is why not every field in $\mathbb{R}^2 \setminus 0$ satisfying (14.57′), as we saw in Example 4, is necessarily a potential field in $\mathbb{R}^2 \setminus 0$.

We now turn from the general description to precise formulations. We begin by stating clearly what we mean we speak of deforming or contracting a path.

Definition 3. A *homotopy* (or *deformation*) in D from a closed path $\gamma_0 : [0,1] \to D$ to a closed path $\gamma_1 : [0,1] \to D$ is a continuous mapping $\Gamma : I^2 \to D$ of the square $I^2 = \{(t^1, t^2) \in \mathbb{R}^2 \big| 0 \le t^i \le 1, i = 1,2\}$ into D such that $\Gamma(t^1, 0) = \gamma_0(t^1)$, $\Gamma(t^1, 1) = \gamma_1(t^1)$, and $\Gamma(0, t^2) = \Gamma(1, t^2)$ for all $t^1, t^2 \in [0,1]$.

Thus a homotopy is a mapping $\Gamma : I^2 \to D$ (Fig. 14.4). If the variable t^2 is regarded as time, according to Definition 3 at each instant of time $t = t^2$ we have a closed path $\Gamma(t^1, t) = \gamma_t$ (Fig. 14.4).[8] The change in this path with time is such that at the initial instant $t = t^2 = 0$ it coincides with γ_0 and at time $t = t^2 = 1$ it becomes γ_1.

Since the condition $\gamma_t(0) = \Gamma(0,t) = \Gamma(1,t) = \gamma_t(1)$, which means that the path γ_t is closed, holds at all times $t \in [0,1]$, the mapping $\Gamma : I^2 \to D$ induces the same mappings $\beta_0(t^1) := \Gamma(t^1, 0) = \Gamma(t^1, 1) =: \beta_1(t^1)$ on the vertical sides of the square I^2.

The mapping Γ is a formalization of our intuitive picture of gradually deforming γ_0 to γ_1.

It is clear that time can be allowed to run backwards, and then we obtain the path γ_0 from γ_1.

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[8] Orienting arrows are shown along certain curves in Fig. 14.4. These arrows will be used a little later; for the time being the reader should not pay any attention to them.

Definition 4. Two closed paths are *homotopic* in a domain if they can be obtained from each other by a homotopy in that domain, that is a homotopy can be constructed in that domain from one to the other.

Remark 2. Since the paths we have to deal with in analysis are as a rule paths of integration, we shall consider only smooth or piecewise-smooth paths and smooth or piecewise-smooth homotopies among them, without noting this explicitly.

For domains in $\mathbb{R}^n$ one can verify that the presence of a continuous homotopy of (piecewise-) smooth paths guarantees the existence of (piecewise-) smooth homotopies of these paths.

Proposition 3. *If the* 1*-form* $\omega^1_{\mathbf{A}}$ *in the domain* D *is such that* $\mathrm{d}\omega^1_{\mathbf{A}} = 0$, *and the closed paths* γ_0 *and* γ_1 *are homotopic in* D, *then*

$$\int_{\gamma_0} \omega^1_{\mathbf{A}} = \int_{\gamma_1} \omega^1_{\mathbf{A}} \, .$$

Proof. Let $\Gamma : I^2 \to D$ be a homotopy from γ_0 to γ_1 (see Fig. 14.4). If I_0 and I_1 are the bases of the square I^2 and J_0 and J_1 its vertical sides, then by definition of a homotopy of closed paths, the restrictions of Γ to I_0 and I_1 coincide with γ_0 and γ_1 respectively, and the restrictions of Γ to J_0 and J_1 give some paths β_0 and β_1 in D. Since $\Gamma(0, t^2) = \Gamma(1, t^2)$, the paths β_0 and β_1 are the same. As a result of the change of variables $x = \Gamma(t)$, the form $\omega^1_{\mathbf{A}}$ transfers to the square I^2 as some 1-form $\omega = \Gamma^* \omega^1_{\mathbf{A}}$. In the process $\mathrm{d}\omega = \mathrm{d}\Gamma^* \omega^1_{\mathbf{A}} = \Gamma^* \mathrm{d}\omega^1_{\mathbf{A}} = 0$, since $\mathrm{d}\omega^1_{\mathbf{A}} = 0$. Hence, by Stokes' formula

$$\int_{\partial I^2} \omega = \int_{I^2} \mathrm{d}\omega = 0 \, .$$

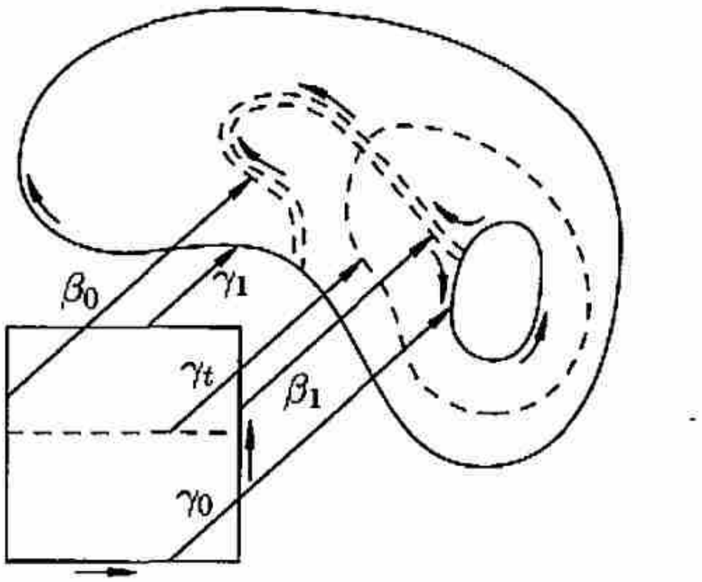

Fig. 14.4.

But

$$\int_{\partial I^2} \omega = \int_{I_0} \omega + \int_{J_1} \omega - \int_{I_1} \omega - \int_{J_0} \omega =$$

$$= \int_{\gamma_0} \omega^1_{\mathbf{A}} + \int_{\beta_1} \omega^1_{\mathbf{A}} - \int_{\gamma_2} \omega^1_{\mathbf{A}} - \int_{\beta_0} \omega^1_{\mathbf{A}} = \int_{\gamma_0} \omega^1_{\mathbf{A}} - \int_{\gamma_1} \omega^1_{\mathbf{A}} . \ \square$$

Definition 5. A domain is *simply connected* if every closed path in it is homotopic to a point (that is, a constant path).

Thus simply connected domains are those in which every closed path can be contracted to a point.

Proposition 4. *If a field* $\mathbf{A}$ *defined in a simply connected domain* D *satisfies the necessary condition* (14.57) *or* (14.57′) *to be a potential field, then it is a potential field in* D.

Proof. By Proposition 1 and Remark 1 it suffices to verify that Eq. (14.59) holds for every smooth path γ in D. The path γ is by hypothesis homotopic to a constant path whose support consists of a single point. The integral over such a one-point path is obviously zero. But by Proposition 3 the integral does not change under a homotopy, and so Eq. (14.59) must hold for γ. $\square$

Remark 3. Proposition 4 subsumes Proposition 2. However, since we had certain applications in mind, we considered it useful to give an independent constructive proof of Proposition 2.

Remark 4. Proposition 2 was proved without invoking the possibility of a smooth homotopy of smooth paths.

14.3.5 Vector Potential. Exact and Closed Forms

Definition 6. A field $\mathbf{A}$ is a *vector potential* for a field $\mathbf{B}$ in a domain $D \subset \mathbb{R}^3$ if the relation $\mathbf{B} = \operatorname{curl} \mathbf{A}$ holds in the domain.

If we recall the connection between vector fields and forms in the oriented Euclidean space $\mathbb{R}^3$ and also the definition of the curl of a vector field, the relation $\mathbf{B} = \operatorname{curl} \mathbf{A}$ can be rewritten as $\omega^2_{\mathbf{B}} = d\omega^1_{\mathbf{A}}$. It follows from this relation that $\omega^3_{\operatorname{div} \mathbf{B}} = d\omega^2_{\mathbf{B}} = d^2\omega^1_{\mathbf{A}} = 0$. Thus we obtain the necessary condition

$$\operatorname{div} \mathbf{B} = 0 , \tag{14.61}$$

which the field $\mathbf{B}$ must satisfy in D in order to have a vector potential, that is, in order to be the curl of a vector field $\mathbf{A}$ in that domain.

A field satisfying condition (14.61) is often, especially in physics, called a *solenoidal field.*

Example 5. In Sect. 14.1 we wrote out the system of Maxwell equations. The second equation of this system is exactly Eq. (14.61). Thus, the desire naturally arises to regard a magnetic field $\mathbf{B}$ as the curl of some vector field $\mathbf{A}$ – the vector potential of $\mathbf{B}$. When solving the Maxwell equations, one passes to exactly such a vector potential.

As can be seen from Definitions 1 and 6, the questions of the scalar and vector potential of vector fields (the latter question being posed only in $\mathbb{R}^3$) are special cases of the general question as to when a differential p-form ω^p is the differential $\mathrm{d}\omega^{p-1}$ of some form ω^{p-1}.

Definition 7. A differential form ω^p is *exact* in a domain D if there exists a form ω^{p-1} in D such that $\omega^p = \mathrm{d}\omega^{p-1}$.

If the form ω^p is exact in D, then $\mathrm{d}\omega^p = \mathrm{d}^2\omega^{p-1} = 0$. Thus the condition

$$\mathrm{d}\omega = 0 \tag{14.62}$$

is a necessary condition for the form ω to be exact.

As we have already seen (Example 4), not every form satisfying this condition is exact. For that reason we make the following definition.

Definition 8. The differential form ω is *closed* in a domain D if it satisfies condition (14.62) there.

The following theorem holds.

Theorem. (Poincaré's lemma.) *If a form is closed in a ball, then it is exact there.*

Here we are talking about a ball in $\mathbb{R}^n$ and a form of any order, so that Proposition 2 is an elementary special case of this theorem.

The Poincaré lemma can also be interpreted as follows: The necessary condition (14.62) for a form to be exact is also locally sufficient, that is, every point of a domain in which (14.62) holds has a neighborhood in which ω is exact.

In particular, if a vector field $\mathbf{B}$ satisfies condition (14.61), it follows from the Poincaré lemma that at least locally it is the curl of some vector field $\mathbf{A}$.

We shall not take the time at this point to prove this important theorem (those who wish to do so can read it in Chap. 15). We prefer to conclude by explaining in general outline the connection between the problem of the exactness of closed forms and the topology of their domains of definition (based on information about 1-forms).

Example 6. Consider the plane $\mathbb{R}^2$ with two points p_1 and p_2 removed (Fig. 14.5), and the paths γ_0, γ_1, and γ_2 whose supports are shown in the figure. The path γ_2 can be contracted to a point inside D, and therefore if a closed form ω is given in D, its integral over γ_2 is zero. The path γ_0 cannot

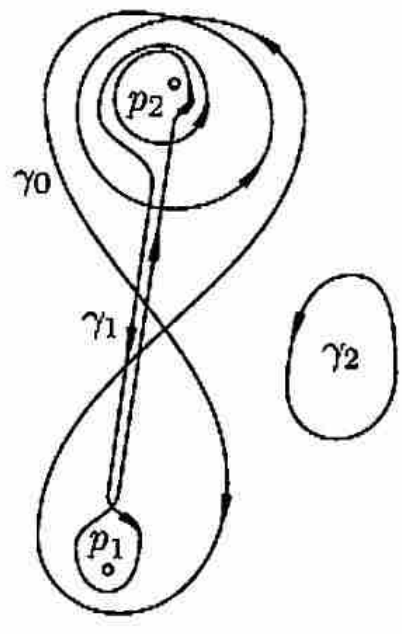

Fig. 14.5.

be contracted to a point, but without changing the value of the integral of the form, it can be homotopically converted into the path γ_1.

The integral over γ_1 obviously reduces to the integral over one cycle enclosing the point p_1 clockwise and the double of the integral over a cycle enclosing p_2 counterclockwise. If T_1 and T_2 are the integrals of the form ω over small circles enclosing the points p_1 and p_2 and traversed, say, counterclockwise, one can see that the integral of the form ω over any closed path in D will be equal to $n_1T_1 + n_2T_2$, where n_1 and n_2 are certain integers indicating how many times we have encircled each of the holes p_1 and p_2 in the plane $\mathbb{R}^2$ and in which direction.

Circles c_1 and c_2 enclosing p_1 and p_2 serve as a sort of basis in which every closed path $\gamma \subset D$ has the form $\gamma = n_1c_1 + n_2c_2$, up to a homotopy, which has no effect on the integral. The quantities $\int\limits_{c_i} \omega = T_i$ are called the *cyclic constants* or the *periods* of the integral. If the domain is more complicated and there are k independent elementary cycles, then in agreement with the expansion $\gamma = n_1c_1 + \cdots + n_kc_k$, it results that $\int\limits_{\gamma} \omega = n_1T_1 + \cdots + n_kT_k$. It turns out that for any set $T_1, \dots, T_k$ of numbers in such a domain one can construct a closed 1-form that will have exactly that set of periods. (This is a special case of de Rham's theorem – see Chap. 15.)

For the sake of visualization, we have resorted here to considering a plane domain, but everything that has been said can be repeated for any domain $D \subset \mathbb{R}^n$.

Example 7. In an anchor ring (the solid domain in $\mathbb{R}^3$ enclosed by a torus) all closed paths are obviously homotopic to a circle that encircles the hole a certain number of times. This circle serves as the unique non-constant basic cycle c.

Moreover, everything that has just been said can be repeated for paths of higher dimension. If instead of one-dimensional closed paths – mappings of a circle or, what is the same, mappings of the one-dimensional sphere – we take mappings of a k-dimensional sphere, introduce the concept of homotopy for them, and examine how many mutually nonhomotopic mappings of the k-dimensional sphere into a given domain $D \subset \mathbb{R}^n$ exist, the result is a certain characteristic of the domain D which is formalized in topology as the so-called kth homotopy group of D and denoted $\pi_k(D)$. If all the mappings of the k-dimensional sphere into D are homotopic to a constant mapping, the group $\pi_k(D)$ is considered trivial. (It consists of the identity element alone.) It can happen that $\pi_1(D)$ is trivial and $\pi_2(D)$ is not.

Example 8. If D is taken to be the space $\mathbb{R}^3$ with the point 0 removed, obviously every closed path in D can be contracted to a point, but a sphere enclosing the point 0 cannot be homotopically converted to a point.

It turns out that the homotopy group $\pi_k(D)$ has less to do with the periods of a closed k-form than the so-called *homology group* $H_k(D)$. (See Chap. 15.)

Example 9. From what has been said we can conclude that, for example, in the domain $\mathbb{R}^3 \setminus 0$ every closed 1-form is exact ($\mathbb{R}^3 \setminus 0$ is a simply connected domain), but not very closed 2-form is exact. In the language of vector fields, this means that every irrotational field $\mathbf{A}$ in $\mathbb{R}^3 \setminus 0$ is the gradient of a function, but not every source-free field $\mathbf{B}$ ($\operatorname{div} \mathbf{B} = 0$) is the curl of some field in this domain.

Example 10. To balance Example 9 we take the anchor ring. For the anchor ring the group $\pi_1(D)$ is not trivial (see Example 7), but $\pi_2(D)$ is trivial, since every mapping $f : S^2 \to D$ of the two-sphere into D can be contracted to a constant mapping (any image of a sphere can be contracted to a point). In this domain not every irrotational field is a potential field, but every source-free field is the curl of some field.

14.3.6 Problems and Exercises

1. Show that every central field $\mathbf{A} = f(r)\mathbf{r}$ is a potential field.

2. Let $\mathbf{F} = -\operatorname{grad} U$ be a potential force field. Show that the stable equilibrium positions of a particle in such a field are the minima of the potential U of that field.

3. For an electrostatic field $\mathbf{E}$ the Maxwell equations (formula (14.12) of Sect. 14.1), as already noted, reduce to the pair of equations $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ and $\nabla \times \mathbf{E} = \mathbf{0}$.

The condition $\nabla \times \mathbf{E} = \mathbf{0}$ means, at least locally, that $\mathbf{E} = -\operatorname{grad} \varphi$. The field of a point charge is a potential field, and since every electric field is the sum (or integral) of such fields, it is always a potential field. Substituting $\mathbf{E} = -\nabla\varphi$ in the first equation of the electrostatic field, we find that its potential satisfies

Poisson's equation[9] $\Delta\varphi = \frac{\rho}{\varepsilon_0}$. The potential φ determines the field completely, so that describing $\mathbf{E}$ reduces to finding the function φ, the solution of the Poisson equation.

Knowing the potential of a point charge (Example 2), solve the following problem.

a) Two charges $+q$ and $-q$ are located at the points $(0, 0, -d/2)$ and $(0, 0, d/2)$ in $\mathbb{R}^3$ with Cartesian coordinates (x, y, z). Show that at distances that are large relative to d the potential of the electrostatic field has the form

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{z}{r^3}qd + o\left(\frac{1}{r^3}\right),$$

where r is the absolute value of the radius-vector $\mathbf{r}$ of the point (x, y, z).

b) Moving very far away from the charges is equivalent to moving the charges together, that is, decreasing the distance d. If we now fix the quantity $qd =: p$ and decrease d, then in the limit we obtain the function $\varphi = \frac{1}{4\pi\varepsilon_0}\frac{z}{r^3}p$ in the domain $\mathbb{R}^3 \setminus 0$. It is convenient to introduce the vector $\mathbf{p}$ equal to p in absolute value and directed from $-q$ to $+q$. We call the pair of charges $-q$ and $+q$ and the construction obtained by the limiting procedure just described a *dipole*, and the vector $\mathbf{p}$ the *dipole moment*. The function φ obtained in the limit is called the *dipole potential*. Find the asymptotics of the dipole potential as one moves away from the dipole along a ray forming angle θ with the direction of the dipole moment.

c) Let φ_0 be the potential of a unit point charge and φ_1 the dipole potential having dipole moment $\mathbf{p}_1$. Show that $\varphi_1 = -(\mathbf{p}_1 \cdot \nabla)\varphi_0$.

d) We can repeat the construction with the limiting passage that we carried out for a pair of charges in obtaining the dipole for the case of four charges (more precisely, for two dipoles with moments $\mathbf{p}_1$ and $\mathbf{p}_2$) and obtain a *quadrupole* and a corresponding potential. In general we can obtain a *multipole of order j* with potential $\varphi_j = (-1)^j(\mathbf{p}_j \cdot \nabla)(\mathbf{p}_{j-1} \cdot \nabla)\cdots(\mathbf{p}_1 \cdot \nabla)\varphi_0 = \sum\limits_{i+k+l=j} Q^j_{ikl}\frac{\partial^j \varphi_0}{\partial x^i \partial y^k \partial z^l}$, where Q^j_{ikl} are the so-called *components of the multipole moment*. Carry out the computations and verify the formula for the potential of a multipole in the case of a quadrupole.

e) Show that the main term in the asymptotics of the potential of a cluster of charges with increasing distance from the cluster is $\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$, where Q is the total charge of the cluster.

f) Show that the main term of the asymptotics of the potential of an electrically neutral body consisting of charges of opposite signs (for example, a molecule) at a distance that is large compared to the dimensions of the body is $\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{e}_r}{r^2}$. Here $\mathbf{e}_r$ is a unit vector directed from the body to the observer; $\mathbf{p} = \sum q_i\mathbf{d}_i$, where q_i is the magnitude of the ith charge and $\mathbf{d}_i$ is its radius-vector. The origin is chosen at some point of the body.

g) The potential of any cluster of charges at a great distance from the cluster can be expanded (asymptotically) in functions of multipole potential type. Show this using the example of the first two terms of such a potential (see d), e), and f)).

[9] S.D. Poisson (1781–1849) – French scientist, specializing in mechanics and physics; his main work was on theoretical and celestial mechanics, mathematical physics, and probability theory. The Poisson equation arose in his research into gravitational potential and attraction by spheroids.

4. Determine whether the following domains are simply connected.

a) the disk $\{(x,y) \in \mathbb{R}^2 \big| x^2+y^2<1\}$;

b) the disk with its center removed $\{(x,y) \in \mathbb{R}^2 \big| 0<x^2+y^2<1\}$.;

c) a ball with its center removed $\{(x,y,z) \in \mathbb{R}^3 \big| 0<x^2+y^2+z^2<1\}$;

d) an annulus $\left\{(x,y) \in \mathbb{R}^2 \Big| \frac{1}{2}<x^2+y^2<1\right\}$;

e) a spherical annulus $\left\{(x,y,z) \in \mathbb{R}^3 \Big| \frac{1}{2}<x^2+y^2+z^2<1\right\}$;

f) an anchor ring in $\mathbb{R}^3$.

5. a) Give the definition of homotopy of paths with endpoints fixed.

b) Prove that a domain is simply connected if and only if every two paths in it having common initial and terminal points are homotopic in the sense of the definition given in part a).

6. Show that

a) every continuous mapping $f: S^1 \to S^2$ of a circle S^1 (a one-dimensional sphere) into a two-dimensional sphere S^2 can be contracted in S^2 to a point (a constant mapping);

b) every continuous mapping $f: S^2 \to S^1$ is also homotopic to a single point;

c) every mapping $f: S^1 \to S^1$ is homotopic to a mapping $\varphi \mapsto n\varphi$ for some $n \in \mathbb{Z}$, where φ is the polar angle;

d) every mapping of the sphere S^2 into an anchor ring is homotopic to a mapping to a single point;

e) every mapping of a circle S^1 into an anchor ring is homotopic to a closed path encircling the hole in the anchor ring n times, for some $n \in \mathbb{Z}$.

7. In the domain $\mathbb{R}^3 \setminus 0$ (three-dimensional space with the point 0 removed) construct:

a) a closed but not exact 2-form;

b) a source-free vector field that is not the curl of any vector field in that domain.

8. a) Can there be closed, but not exact forms of degree $p<n-1$ in the domain $D=\mathbb{R}^n \setminus 0$ (the space $\mathbb{R}^n$ with the point 0 removed)?

b) Construct a closed but not exact form of degree $p=n-1$ in $D=\mathbb{R}^n \setminus 0$.

9. If a 1-form ω is closed in a domain $D \subset \mathbb{R}^n$, then by Proposition 2 every point $x \in D$ has a neighborhood $U(x)$ inside which ω is exact. From now on ω is assumed to be a closed form.

a) Show that if two paths $\gamma_i: [0,1] \to D$, $i=1,2$, have the same initial and terminal points and differ only on an interval $[\alpha,\beta] \subset [0,1]$ whose image under either of the mappings γ_i is contained inside the same neighborhood $U(x)$, then $\int\limits_{\gamma_1} \omega = \int\limits_{\gamma_2} \omega$.

b) Show that for every path $[0,1] \ni t \mapsto \gamma(t) \in D$ one can find a number $\delta > 0$ such that if the path $\widetilde{\gamma}$ has the same initial and terminal point as γ and differs from γ at most by δ, that is $\max_{0\le t\le 1} |\widetilde{\gamma}(t) - \gamma(t)| \le \delta$, then $\int_{\widetilde{\gamma}} \omega = \int_{\gamma} \omega$.

c) Show that if two paths γ_1 and γ_2 with the same initial and terminal points are homotopic in D as paths with fixed endpoints, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ for any closed form ω in D.

10. a) It will be proved below that every continuous mapping $\Gamma : I^2 \to D$ of the square I^2 can be uniformly approximated with arbitrary accuracy by a smooth mapping (in fact by a mapping with polynomial components). Deduce from this that if the paths γ_1 and γ_2 in the domain D are homotopic, then for every $\varepsilon > 0$ there exist smooth mutually homotopic paths $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ such that $\max_{0\le t\le 1} |\widetilde{\gamma}_i(t) - \gamma_i(t)| \le \varepsilon$, $i = 1, 2$.

b) Using the results of Example 9, show now that if the integrals of a closed form in D over smooth homotopic paths are equal, then they are equal for any paths that are homotopic in this domain (regardless of the smoothness of the homotopy). The paths themselves, of course, are assumed to be as regular as they need to be for integration over them.

11. a) Show that if the forms ω^p, ω^{p-1}, and $\widetilde{\omega}^{p-1}$ are such that $\omega^p = d\omega^{p-1} = d\widetilde{\omega}^{p-1}$, then (at least locally) one can find a form ω^{p-2} such that $\widetilde{\omega}^{p-1} = \omega^{p-1} + d\omega^{p-2}$. (The fact that any two forms that differ by the differential of a form have the same differential obviously follows from the relation $d^2\omega = 0$.)

b) Show that the potential φ of an electrostatic field (Problem 3) is determined up to an additive constant, which is fixed if we require that the potential tend to zero at infinity.

12. The Maxwell equations (formula (14.12) of Sect. 14.1) yield the following pair of magnetostatic equations: $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = -\frac{\mathbf{j}}{\varepsilon_0 c^2}$. The first of these shows that at least locally, $\mathbf{B}$ has a vector potential $\mathbf{A}$, that is, $\mathbf{B} = \nabla \times \mathbf{A}$.

a) Describe the amount of arbitrariness in the choice of the potential $\mathbf{A}$ of the magnetic field $\mathbf{B}$ (see Problem 11a)).

b) Let x, y, z be Cartesian coordinates in $\mathbb{R}^3$. Find potentials $\mathbf{A}$ for a uniform magnetic field $\mathbf{B}$ directed along the z-axis, each satisfying one of the following additional requirements: the field $\mathbf{A}$ msut have the form $(0, A_y, 0)$; the field $\mathbf{A}$ must have the form $(A_x, 0, 0)$; the field $\mathbf{A}$ must have the form $(A_x, A_y, 0)$; the field $\mathbf{A}$ must be invariant under rotations about the z-axis.

c) Show that the choice of the potential $\mathbf{A}$ satisfying the additional requirement $\nabla \cdot \mathbf{A} = 0$ reduces to solving Poisson's equation; more precisely, to finding a scalar-valued function ψ satisfying the equation $\Delta\psi = f$ for a given scalar-valued function f.

d) Show that if the potential $\mathbf{A}$ of a static magnetic field $\mathbf{B}$ is chosen so that $\nabla \cdot \mathbf{A} = 0$, it will satisfy the vector Poisson equation $\Delta \mathbf{A} = -\frac{\mathbf{j}}{\varepsilon_0 c^2}$. Thus, invoking the potential makes it possible to reduce the problem of finding electrostatic and magnetostatic fields to solving Poisson's equation.

13. The following *theorem of Helmholtz*[10] is well known: *Every smooth field* $\mathbf{F}$ *in a domain* D *of oriented Euclidean space* $\mathbb{R}^3$ *can be decomposed into a sum* $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ *of an irrotational field* $\mathbf{F}_1$ *and a solenoidal field* $\mathbf{F}_2$. Show that the construction of such a decomposition can be reduced to solving a certain Poisson equation.

14. Suppose a given mass of a certain substance passes from a state characterized thermodynamically by the parameters V_0, P_0 (T_0) into the state V, P, (T). Assume that the process takes place slowly (quasi-statically) and over a path γ in the plane of states (with coordinates V, P). It can be proved in thermodynamics that the quantity $S = \int\limits_\gamma \frac{\delta Q}{T}$, where δQ is the heat exchange form, depends only on the initial point (V_0, P_0) and the terminal point (V, P) of the path, that is, after one of these points is fixed, for example (V_0, P_0), S becomes a function of the state (V, P) of the system. This function is called the *entropy* of the system.

a) Deduce from this that the form $\omega = \frac{\delta Q}{T}$ is exact, and that $\omega = \mathrm{d}S$.

b) Using the form of δQ given in Problem 6 of Sect. 13.1 for an ideal gas, find the entropy of an ideal gas.

14.4 Examples of Applications

To show the concepts we have introduced in action, and also to explain the physical meaning of the Gauss–Ostrogradskii–Stokes formula as a conservation law, we shall examine here some illustrative and important equations of mathematical physics.

14.4.1 The Heat Equation

We are studying the scalar field $T = T(x, y, z, t)$ of the temperature of a body being observed as a function of the point (x, y, z) of the body and the time t. As a result of heat transfer between various parts of the body the field T may vary. However, this variation is not arbitrary; it is subject to a particular law which we now wish to write out explicitly.

Let D be a certain three-dimensional part of the observed body bounded by a surface S. If there are no heat sources inside S, a change in the internal energy of the substance in D can occur only as the result of heat transfer, that is, in this case by the transfer of energy across the boundary S of D.

By computing separately the variation in internal energy in the volume D and the flux of energy across the surface S, we can use the law of conservation of energy to equate these two quantities and obtain the needed relation.

It is known that an increase in the temperature of a homogeneous mass m by ΔT requires energy $cm\Delta T$, where c is the specific heat capacity of

[10] H.L.F. Helmholtz (1821–1894) – German physicist and mathematician; one of the first to discover the general law of conservation of energy. Actually, he was the first to make a clear distinction between the concepts of force and energy.

the substance under consideration. Hence if our field T changes by $\Delta T = T(x, y, z, t + \Delta t) - T(x, y, z, t)$ over the time interval Δt, the internal energy in D will have changed by an amount

$$\iiint\limits_D c\rho \Delta T \,\mathrm{d}V , \tag{14.63}$$

where $\rho = \rho(x, y, z)$ is the density of the substance.

It is known from experiments that over a wide range of temperatures the quantity of heat flowing across a distinguished area $\mathrm{d}\boldsymbol{\sigma} = \mathbf{n}\,\mathrm{d}\sigma$ per unit time as the result of heat transfer is proportional to the flux $-\mathrm{grad}\, T \cdot \mathrm{d}\boldsymbol{\sigma}$ of the field $-\mathrm{grad}\, T$ across that area (the gradient is taken with respect to the spatial variables x, y, z). The coefficient of proportionality k depends on the substance and is called its *coefficient of thermal conductivity*. The negative sign in front of $\mathrm{grad}\, T$ corresponds to the fact that the energy flows from hotter parts of the body to cooler parts. Thus, the energy flux (up to terms of order $o(\Delta t)$)

$$\Delta t \iint\limits_S -k\, \mathrm{grad}\, T \cdot \mathrm{d}\boldsymbol{\sigma} \tag{14.64}$$

takes place across the boundary S of D in the direction of the external normal over the time interval Δt.

Equating the quantity (14.63) to the negative of the quantity (14.64), dividing by Δt, and passing to the limit as $\Delta t \to 0$, we obtain

$$\iiint\limits_D c\rho \frac{\partial T}{\partial t} \,\mathrm{d}V = \iint\limits_S k\, \mathrm{grad}\, T \cdot \mathrm{d}\boldsymbol{\sigma} . \tag{14.65}$$

This equality is the equation for the function T. Assuming T is sufficiently smooth, we transform (14.65) using the Gauss–Ostrogradskii formula:

$$\iiint\limits_D c\rho \frac{\partial T}{\partial t} \,\mathrm{d}V = \iiint\limits_D \mathrm{div}\,(k\, \mathrm{grad}\, T) \,\mathrm{d}V .$$

Hence, since D is arbitrary, it follows obviously that

$$c\rho \frac{\partial T}{\partial t} = \mathrm{div}\,(k\, \mathrm{grad}\, T) . \tag{14.66}$$

We have now obtained the differential version of the integral equation (14.65).

If there were heat sources (or sinks) in D whose intensities have density $F(x, y, z, t)$, instead of (14.65) we would write the equality

$$\iiint\limits_D c\rho \frac{\partial T}{\partial t} \,\mathrm{d}V = \iint\limits_S k\, \mathrm{grad}\, T \cdot \mathrm{d}\boldsymbol{\sigma} + \iiint\limits_D F \,\mathrm{d}V , \tag{14.65'}$$

and then instead of (14.66) we would have the equation

$$c\rho\frac{\partial T}{\partial t} = \operatorname{div}(k \operatorname{grad} T) + F \,. \tag{14.66'}$$

If the body is assumed isotropic and homogeneous with respect to its heat conductivity, the coefficient k in (14.66) will be constant, and the equation will transform to the canonical form

$$\frac{\partial T}{\partial t} = a^2 \Delta T + f \,, \tag{14.67}$$

where $f = \frac{F}{c\rho}$ and $a^2 = \frac{k}{c\rho}$ is the *coefficient of thermal diffusivity.* The equation (14.67) is usually called the *heat equation.*

In the case of steady-state heat transfer, in which the field T is independent of time, this equation becomes *Poisson's equation*

$$\Delta T = \varphi \,, \tag{14.68}$$

where $\varphi = -\frac{1}{a^2} f$; and if in addition there are no heat sources in the body, the result is *Laplace's equation*

$$\Delta T = 0 \,. \tag{14.69}$$

The solutions of Laplace's equation, as already noted, are called *harmonic functions.* In the thermophysical interpretation, harmonic functions correspond to steady-state temperature fields in a body in which the heat flows occur without any sinks or sources inside the body itself, that is, all sources are located outside the body. For example, if we maintain a steady temperature distribution $T\big|_{\partial V} = \tau$ over the boundary ∂V of a body, then the temperature field in the body V will eventually stabilize in the form of a harmonic function T. Such an interpretation of the solutions of the Laplace equation (14.69) enables us to predict a number of properties of harmonic functions. For example, one must presume that a harmonic function in V cannot have local maxima inside the body; otherwise heat would only flow away from these hotter portions of the body, and they would cool off, contrary to the assumption that the field is stationary.

14.4.2 The Equation of Continuity

Let $\rho = \rho(x, y, z, t)$ be the density of a material medium that fills a space being observed and $\mathbf{v} = \mathbf{v}(x, y, z, t)$ the velocity field of motion of the medium as function of the point of space (x, y, z) and the time t.

From the law of conservation of mass, using the Gauss–Ostrogradskii formula, we can find an interconnection between these quantities.

Let D be a domain in the space being observed bounded by a surface S. Over the time interval Δt the quantity of matter in D varies by an amount

$$\iiint\limits_D \big(\rho(x,y,z,t+\Delta t) - \rho(x,y,z,t)\big)\,\mathrm{d}V\ .$$

Over this small time interval Δt, the flow of matter across the surface S in the direction of the outward normal to S is (up to $o(\Delta t)$)

$$\Delta t \cdot \iint\limits_S \rho\mathbf{v}\cdot \mathrm{d}\boldsymbol{\sigma}\ .$$

If there were no sources or sinks in D, then by the law of conservation of matter, we would have

$$\iiint\limits_D \Delta\rho\,\mathrm{d}V = -\Delta t \iint\limits_S \rho\mathbf{v}\cdot \mathrm{d}\boldsymbol{\sigma}$$

or, in the limit as $\Delta t \to 0$

$$\iiint\limits_D \frac{\partial \rho}{\partial t}\,\mathrm{d}V = -\iint\limits_S \rho\mathbf{v}\cdot \mathrm{d}\boldsymbol{\sigma}\ .$$

Applying the Gauss–Ostrogradskii formula to the right-hand side of this equality and taking account of the fact that D is an arbitrary domain, we conclude that the following relation must hold for sufficiently smooth functions ρ and $\mathbf{v}$:

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho\mathbf{v})\ , \tag{14.70}$$

called the *equation of continuity* of a continuous medium.

In vector notation the equation of continuity can be written as

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\ , \tag{14.70'}$$

or, in more expanded form,

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0\ , \tag{14.70''}$$

If the medium is incompressible (a liquid), the volumetric outflow of the medium across a closed surface S must be zero:

$$\iint\limits_S \mathbf{v}\cdot \mathrm{d}\boldsymbol{\sigma} = 0\ ,$$

from which (again on the basis of the Gauss–Ostrogradskii formula) it follows that for an incompressible medium

$$\operatorname{div}\mathbf{v} = 0\ . \tag{14.71}$$

Hence, for an incompressible medium of variable density (a mixture of water and oil) Eq. (14.70″) becomes

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0 . \qquad (14.72)$$

If the medium is also homogeneous, then $\nabla \rho = 0$ and therefore $\frac{\partial \rho}{\partial t} = 0$.

14.4.3 The Basic Equations of the Dynamics of Continuous Media

We shall now derive the equations of the dynamics of a continuous medium moving in space. Together with the functions ρ and $\mathbf{v}$ already considered, which will again denote the density and the velocity of the medium at a given point (x, y, z) of space and at a given instant t of time, we consider the pressure $p = p(x, y, z, t)$ as a function of a point of space and time.

In the space occupied by the medium we distinguish a domain D bounded by a surface S and consider the forces acting on the distinguished volume of the medium at a fixed instant of time.

Certain force fields (for example, gravitation) are acting on each element $\rho \, \mathrm{d}V$ of mass of the medium. These fields create the so-called *mass forces*. Let $\mathbf{F} = \mathbf{F}(x, y, z, t)$ be the density of the external fields of mass force. Then a force $\mathbf{F}\rho \, \mathrm{d}V$ acts on the element from the direction of these fields. If this element has an acceleration $\mathbf{a}$ at a given instant of time, then by Newton's second law, this is equivalent to the presence of another mass force called inertia, equal to $-\mathbf{a}\rho \, \mathrm{d}V$.

Finally, on each element $\mathrm{d}\boldsymbol{\sigma} = \mathbf{n} \, \mathrm{d}\sigma$ of the surface S there is a surface tension due to the pressure of the particles of the medium near those in D, and this surface force equals $-p \, \mathrm{d}\boldsymbol{\sigma}$ (where $\mathbf{n}$ is the outward normal to S).

By d'Alembert's principle, at each instant during the motion of of any material system, all the forces applied to it, including inertia, are in mutual equilibrium, that is, the force required to balance them is zero. In our case, this means that

$$\iiint\limits_D (\mathbf{F} - \mathbf{a})\rho \, \mathrm{d}V - \iint\limits_S p \, \mathrm{d}\boldsymbol{\sigma} = \mathbf{0} . \qquad (14.73)$$

The first term in this sum is the equilibrant of the mass and inertial forces, and the second is the equilibrant of the pressure on the surface S bounding the volume. For simplicity we shall assume that we are dealing with an ideal (nonviscous) fluid or gas, in which the pressure on the surface $\mathrm{d}\boldsymbol{\sigma}$ has the form $p \, \mathrm{d}\boldsymbol{\sigma}$, where the number p is independent of the orientation of the area in the space.

Applying formula (14.47) from Sect. 14.2, we find by (14.73) that

$$\iiint\limits_D (\mathbf{F} - \mathbf{a})\rho \, \mathrm{d}V - \iiint\limits_D \operatorname{grad} p \, \mathrm{d}v = \mathbf{0} ,$$

from which, since the domain D is arbitrary, it follows that

$$\rho \mathbf{a} = \rho \mathbf{F} - \operatorname{grad} p\,. \tag{14.74}$$

In this local form the equation of motion of the medium corresponds perfectly to Newton's law of motion for a material particle.

The acceleration $\mathbf{a}$ of a particle of the medium is the derivative $\frac{d\mathbf{v}}{dt}$ of the velocity $\mathbf{v}$ of the particle. If $x = x(t)$, $y = y(t)$, $z = z(t)$ is the law of motion of a particle in space and $\mathbf{v} = \mathbf{v}(x, y, z, t)$ is the velocity field of the medium, then for each individual particle we obtain

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt}$$

or

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\,.$$

Thus the equation of motion (14.74) assumes the following form

$$\frac{d\mathbf{v}}{dt} = \mathbf{F} - \frac{1}{\rho}\operatorname{grad} p \tag{14.75}$$

or

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p\,. \tag{14.76}$$

Equation (14.76) is usually called *Euler's hydrodynamic equation.*

The vector equation (14.76) is equivalent to a system of three scalar equations for the three components of the vector $\mathbf{v}$ and the pair of functions ρ and p.

Thus, Euler's equation does not completely determine the motion of an ideal continuous medium. To be sure, it is natural to adjoin to it the equation of continuity (14.70), but even then the system is underdetermined.

To make the motion of the medium determinate one must also add to Eqs. (14.70) and (14.76) some information on the thermodynamic state of the medium (for example, the equation of state $f(p, \rho, T) = 0$ and the equation for heat transfer.) The reader may obtain some idea of what these relations can yield in the final subsection of this section.

14.4.4 The Wave Equation

We now consider the motion of a medium corresponding to the propagation of an acoustic wave. It is clear that such a motion is also subject to Eq. (14.76); this equation can be simplified due to the specifics of the phenomenon.

Sound is an alternating state of rarefaction and compression of a medium, the deviation of the pressure from its mean value in a sound wave being very small – of the order of 1%. Therefore acoustic motion consists of small deviations of the elements of volume of the medium from the equilibrium position at small velocities. However, the rate of propagation of the disturbance (wave) through the medium is comparable with the mean velocity of motion of the molecules of the medium and usually exceeds the rate of heat transfer between the different parts of the medium under consideration. Thus, an acoustic motion of a volume of gas can be regarded as small oscillations about the equilibrium position occurring without heat transfer (an adiabatic process).

Neglecting the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the equation of motion (14.76) in view of the small size of the macroscopic velocities $\mathbf{v}$, we obtain the equality

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \rho\mathbf{F} - \nabla p\,.$$

If we neglect the term of the form $\frac{\partial\rho}{\partial t}\mathbf{v}$ for the same reason, the last equality reduces to the equation

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) = \rho\mathbf{F} - \nabla p\,.$$

Applying the operator ∇ (on x, y, z coordinates) to it, we obtain

$$\frac{\partial}{\partial t}(\nabla\cdot\rho\mathbf{v}) = \nabla\cdot\rho\mathbf{F} - \Delta p\,.$$

Using the equation of continuity (14.70′) and introducing the notation $\nabla\cdot\rho\mathbf{F} = -\Phi$, we arrive at the equation

$$\frac{\partial^2\rho}{\partial t^2} = \Phi + \Delta p\,. \tag{14.77}$$

If we can neglect the influence of the exterior fields, Eq. (14.77) reduces to the relation

$$\frac{\partial^2\rho}{\partial t^2} = \Delta p \tag{14.78}$$

between the density and pressure in the acoustic medium. Since the process is adiabatic, the equation of state $f(p,\rho,T) = 0$ reduces to a relation $\rho = \psi(p)$, from which it follows that $\frac{\partial^2\rho}{\partial t^2} = \psi'(p)\frac{\partial^2 p}{\partial t^2} + \psi''(p)\left(\frac{\partial p}{\partial t}\right)^2$. Since the pressure oscillations are small in an acoustic wave, one may assume that $\psi'(p) \equiv \psi'(p_0)$, where p_0 is the equilibrium pressure. Then $\psi'' = 0$ and $\frac{\partial^2\rho}{\partial t^2} \approx \psi'(p)\frac{\partial^2 p}{\partial t^2}$. Taking this into account, from (14.78) we finally obtain

$$\frac{\partial^2 p}{\partial t^2} = a^2\Delta p\,, \tag{14.79}$$

where $a = \left(\psi'(p_0)\right)^{-1/2}$. This equation describes the variation in pressure in a medium in a state of acoustic motion. Equation (14.79) describes the simplest wave process in a continuous medium. It is called the *homogeneous wave equation.* The quantity a has a simple physical meaning: it is the speed of propagation of an acoustic disturbance in the medium, that is, the speed of sound in it (see Problem 4).

In the case of forced oscillations, when certain forces are acting on each element of volume of the medium, the three-dimensional density of whose distribution is given, Eq. (14.79) is replaced by the relation

$$\frac{\partial^2 p}{\partial t^2} = a^2 \Delta p + f \tag{14.80}$$

corresponding to Eq. (14.77), which for $f \not\equiv 0$ is called the *inhomogeneous wave equation.*

14.4.5 Problems and Exercises

1. Suppose the velocity field $\mathbf{v}$ of a moving continuous medium is a potential field. Show that if the medium is incompressible, the potential φ of the field $\mathbf{v}$ is a harmonic function, that is, $\Delta\varphi = 0$ (see (14.71).

2. a) Show that Euler's equation (14.76) can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}\left(\frac{1}{2}\mathbf{v}^2\right) - \mathbf{v} \times \operatorname{curl} \mathbf{v} = \mathbf{F} - \frac{1}{\rho}\operatorname{grad} p$$

(see Problem 1 of Sect. 14.1).

b) Verify on the basis of the equation of a) that an irrotational flow ($\operatorname{curl} \mathbf{v} = 0$) of a homogeneous incompressible liquid can occur only in a potential field $\mathbf{F}$.

c) It turns out (Lagrange's theorem) that if at some instant the flow in a potential field $\mathbf{F} = \operatorname{grad} U$ is irrotational, then it always has been and always will be irrotational. Such a flow consequently is at least locally a potential flow, that is, $\mathbf{v} = \operatorname{grad}\varphi$. Verify that for a potential flow of a homogeneous incompressible liquid taking place in a potential field $\mathbf{F}$, the following relation holds at each instant of time:

$$\operatorname{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} - U\right) = 0\,.$$

d) Derive the so-called *Cauchy integral* from the equality just obtained:

$$\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} - U = \Phi(t)\,,$$

a relation that asserts that the left-hand side is independent of the spatial coordinates.

e) Show that if the flow is also steady-state, that is, the field $\mathbf{v}$ is independent of time, the following relation holds

$$\frac{v^2}{2} + \frac{p}{\rho} - U = \text{const}\,,$$

called the *Bernoulli integral.*

3. A flow whose velocity field has the form $\mathbf{v} = (v_x, v_y, 0)$ is naturally called *plane-parallel* or simply a *planar flow*.

a) Show that the conditions $\operatorname{div} \mathbf{v} = 0$, $\operatorname{curl} \mathbf{v} = \mathbf{0}$ for a flow to be incompressible and irrotational have the following forms:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0\,, \quad \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0\,.$$

b) Show that these equations at least locally guarantee the existence of functions $\psi(x, y)$ and $\varphi(x, y)$ such that $(-v_y, v_x) = \operatorname{grad} \psi$ and $(v_x, v_y) = \operatorname{grad} \varphi$.

c) Verify that the level curves $\varphi = c_1$ and $\psi = c_2$ of these functions are orthogonal and show that in the steady-state flow the curves $\psi = c$ coincide with the trajectories of the moving particles of the medium. It is for that reason that the function ψ is called the *current function*, in contrast to the function φ, which is the *velocity potential*.

d) Show, assuming that the functions φ and ψ are sufficiently smooth, that they are both harmonic functions and satisfy the *Cauchy–Riemann equations*:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}\,, \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}\,.$$

Harmonic functions satisfying the Cauchy–Riemann equations are called *conjugate harmonic functions*.

e) Verify that the function $f(z) = (\varphi + i\psi)(x, y)$, where $z = x + iy$, is a differentiable function of the complex variable z. This determines the connection of the planar problems of hydrodynamics with the theory of functions of a complex variable.

4. Consider the elementary version $\frac{\partial^2 p}{\partial t^2} = a^2 \frac{\partial^2 p}{\partial x^2}$ of the wave equation (14.79). This is the case of a plane wave in which the pressure depends only on the x-coordinate of the point (x, y, z) of space.

a) By making the change of variable $u = x - at$, $v = x + at$, reduce this equation to the form $\frac{\partial^2 p}{\partial u \partial v} = 0$ and show that the general form of the solution of the original equation is $p = f(x + at) + g(x - at)$, where f and g are arbitrary functions of class $C^{(2)}$.

b) Interpret the solution just obtained as two waves $f(x)$ and $g(x)$ propagating left and right along the x-axis with velocity a.

c) Assuming that the quantity a is the velocity of propagation of a disturbance even in the general case (14.79), and taking account of the relation $a = \left(\psi'(p_0)\right)^{-1/2}$, find, following Newton, the velocity c_N of sound in air, assuming that the temperature in an acoustic wave is constant, that is, assuming that the process of acoustic oscillation is isothermic. (The equation of state is $\rho = \frac{\mu p}{RT}$, $R = 8.31 \frac{\mathrm{J}}{\mathrm{deg \cdot mole}}$ is the universal gas constant, and $\mu = 28.8 \frac{\mathrm{g}}{\mathrm{mole}}$ is the molecular weight of air. Carry out the computation for air at a temperature of $0°$ C, that is, $T = 273\,\mathrm{K}$. Newton found that $c_N = 280\,\mathrm{m/s}$.)

d) Assuming that the process of acoustic vibrations is adiabatic, find, following Laplace, the velocity c_L of sound in air, and thereby sharpen Newton's result c_N. (In an adiabatic process $p = c\rho^\gamma$. This is Poisson's formula from Problem 6 of

Sect. 13.1. Show that if $c_N = \sqrt{\frac{p}{\rho}}$, then $c_L = \gamma\sqrt{\frac{p}{\rho}}$. For air $\gamma \approx 1.4$. Laplace found $c_L = 330\,\mathrm{m/s}$, which is in excellent agreement with experiment.)

5. Using the scalar and vector potentials one can reduce the Maxwell equations ((14.12) of Sect. 14.1) to the wave equation (more precisely,to several wave equations of the same type). By solving this problem, you will verify this statement.

a) It follows from the equation $\nabla \cdot \mathbf{B} = 0$ that at least locally $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential of the field $\mathbf{B}$.

b) Knowing that $\mathbf{B} = \nabla \times \mathbf{A}$, show that the equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ implies that at least locally there exists a scalar function φ such that $\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$.

c) Verify that the fields $\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ do not change if instead of φ and $\mathbf{A}$ we take another pair of potentials $\widetilde{\varphi}$ and $\widetilde{\mathbf{A}}$ such that $\widetilde{\varphi} = \varphi - \frac{\partial \psi}{\partial t}$ and $\widetilde{\mathbf{A}} = \mathbf{A} + \nabla\psi$, where ψ is an arbitrary function of class $C^{(2)}$.

d) The equation $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ implies the first relation $-\nabla^2\varphi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \frac{\rho}{\partial\varepsilon_0}$ between the potentials φ and $\mathbf{A}$.

e) The equation $c^2\nabla \times \mathbf{B} - \frac{\mathbf{E}}{\partial t} = \frac{\mathbf{j}}{\partial\varepsilon_0}$ implies the second relation

$$-c^2\nabla^2\mathbf{A} + c^2\nabla(\nabla \cdot \mathbf{A}) + \frac{\partial}{\partial t}\nabla\varphi + \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\mathbf{j}}{\varepsilon_0}$$

between the potentials φ and $\mathbf{A}$.

f) Using c), show that by solving the auxiliary wave equation $\Delta\psi + f = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$, without changing the fields $\mathbf{E}$ and $\mathbf{B}$ one can choose the potentials φ and $\mathbf{A}$ so that they satisfy the additional (so-called *gauge*) condition $\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial\varphi}{\partial t}$.

g) Show that if the potentials φ and $\mathbf{A}$ are chosen as stated in f), then the required inhomogeneous wave equations

$$\frac{\partial^2\varphi}{\partial t^2} = c^2\Delta\varphi + \frac{\rho c^2}{\varepsilon_0}, \quad \frac{\partial^2 \mathbf{A}}{\partial t^2} = c^2\Delta\mathbf{A} + \frac{\mathbf{j}}{\varepsilon_0}$$

for the potentials φ and $\mathbf{A}$ follow from d) and e). By finding φ and $\mathbf{A}$, we also find the fields $\mathbf{E} = \nabla\varphi$, $\mathbf{B} = \nabla \times \mathbf{A}$.

15 *Integration of Differential Forms on Manifolds

15.1 A Brief Review of Linear Algebra

15.1.1 The Algebra of Forms

Let X be a vector space and $F^k : X^k \to \mathbb{R}$ a real-valued k-form on X. If $e_1, \dots, e_n$ is a basis in X and $x_1 = x^{i_1}e_{i_1}, \dots, x_k = x^{i_k}e_{i_k}$ is the expansion of the vectors $x_1, \dots, x_k \in X$ with respect to this basis, then by the linearity of F^k, with respect to each argument

$$F^k(x_1, \dots, x_k) = F^k(x^{i_1}e_{i_1}, \dots, x^{i_k}e_{i_k}) = \\ = F^k(e_{i_1}, \dots, e_{i_k})x^{i_1} \cdot \ldots \cdot x^{i_k} = a_{i_1 \dots i_k} x^{i_1} \cdot \ldots \cdot x^{i_k} . \quad (15.1)$$

Thus, after a basis is given in X, one can identify the k-form $F^k : X^k \to \mathbb{R}$ with the set of numbers $a_{i_1 \dots i_k} = F^k(e_{i_1}, \dots, e_{i_k})$.

If $\widetilde{e}_1, \dots, \widetilde{e}_n$ is another basis in X and $\widetilde{a}_{j_1 \dots j_k} = F^k(\widetilde{e}_{j_1}, \dots, \widetilde{e}_{j_k})$, then, setting $\widetilde{e}_j = c^i_j e_i$, $j = 1, \dots, n$, we find the (tensor) law

$$\widetilde{a}_{j_1 \dots j_k} = F^k(c^{i_1}_{j_1} e_{i_1}, \dots, c^{i_k}_{j_k} e_{i_k}) = a_{i_1 \dots i_k} c^{i_1}_{j_1} \cdot \ldots \cdot c^{i_k}_{j_k} \quad (15.2)$$

for transformation of the number sets $a_{i_1 \dots i_k}$, $\widetilde{a}_{j_1 \dots j_k}$ corresponding to the same form F^k.

The set $\mathcal{F}^k := \{F^k : X^k \to \mathbb{R}\}$ of k-forms on a vector space X is itself a vector space relative to the standard operations

$$(F^k_1 + F^k_2)(x) := F^k_1(x) + F^k_2(x) , \quad (15.3)$$

$$(\lambda F^k)(x) := \lambda F^k(x) \quad (15.4)$$

of addition of k-forms and multiplication of a k-form by a scalar.

For forms F^k and F^l of arbitrary degrees k and l the following *tensor product* operation $\otimes$ is defined:

$$(F^k \otimes F^l)(x_1, \dots, x_k, x_{k+1}, \dots, x_{k+l}) := \\ = F^k(x_1, \dots, x_k) F^l(x_{k+1}, \dots, x_{k+l}) . \quad (15.5)$$

Thus $F^k \otimes F^l$ is a form F^{k+l} of degree $k + l$. The following relations are obvious:

$$(\lambda F^k) \otimes F^l = \lambda(F^k \otimes F^l)\,, \tag{15.6}$$
$$(F_1^k + F_2^k) \otimes F^l = F_1^k \otimes F^l + F_2^k \otimes F^l\,, \tag{15.7}$$
$$F^k \otimes (F_1^l + F_2^l) = F^k \otimes F_1^l + F^k \otimes F_2^l\,, \tag{15.8}$$
$$(F^k \otimes F^l) \otimes F^m = F^k \otimes (F^l \otimes F^m)\,. \tag{15.9}$$

Thus the set $\mathcal{F} = \{\mathcal{F}^k\}$ of forms on the vector space X is a graded algebra $\mathcal{F} = \bigotimes\limits_k \mathcal{F}^k$ with respect to these operations, in which the vector-space operations are carried out inside each space $\mathcal{F}^k$ occurring in the direct sum, and if $F^k \in \mathcal{F}^k$, $F^l \in \mathcal{F}^l$, then $F^k \otimes F^l \in \mathcal{F}^{k+l}$.

Example 1. Let X^* be the dual space to X (consisting of the linear functionals on X) and $e^1, \dots, e^n$ the basis of X^* dual to the basis $e_1, \dots, e_n$ in X, that is, $e^i(e_j) = \delta^i_j$.

Since $e^i(x) = e^i(x^j e_j) = x^j e^i(e_j) = x^j \delta^i_j = x^i$, taking account of (15.1) and (15.9), we can write any k-form $F^k : X^k \to \mathbb{R}$ as

$$F^k = a_{i_1\dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}\,. \tag{15.10}$$

15.1.2 The Algebra of Skew-symmetric Forms

Let us now consider the space Ω^k of skew-symmetric forms in $\mathcal{F}^k$, that is, $\omega \in \Omega^k$ if the equality

$$\omega(x_1, \dots, x_i, \dots, x_j, \dots, x_k) = -\omega(x_1, \dots, x_j, \dots, x_i, \dots, x_k)$$

holds for any distinct indices $i, j \in \{1, \dots, n\}$.

From any form $F^k \in \mathcal{F}^k$ one can obtain a skew-symmetric form using the operation $A : \mathcal{F} \to \Omega^k$ of *alternation*, defined by the relation

$$AF^k(x_1, \dots, x_k) := \frac{1}{k!} F^k(x_{i_1}, \dots, x_{i_k}) \delta^{i_1 \dots i_k}_{1 \dots k}\,, \tag{15.11}$$

where

$$\delta^{i_1\dots i_k}_{1\dots k} = \begin{cases} 1, & \text{if the permutation } \begin{pmatrix} i_1 \dots i_k \\ 1 \dots k \end{pmatrix} \text{ is even}\,, \\ -1, & \text{if the permutation } \begin{pmatrix} i_1 \dots i_k \\ 1 \dots k \end{pmatrix} \text{ is odd}\,, \\ 0, & \text{if } \begin{pmatrix} i_1 \dots i_k \\ 1 \dots k \end{pmatrix} \text{ is not a permutation}\,. \end{cases}$$

If F^k is a skew-symmetric form, then, as one can see from (15.11), $AF^k = F^k$. Thus $A(AF^k) = AF^k$ and $A\omega = \omega$ if $\omega \in \Omega^k$. Hence $A : \mathcal{F}^k \to \Omega^k$ is a mapping of $\mathcal{F}^k$ *onto* Ω^k.

Comparing Definitions (15.3), (15.4), and (15.11), we obtain

$$A(F_1^k + F_2^k) = AF_1^k + AF_2^k\,, \tag{15.12}$$
$$A(\lambda F^k) = \lambda AF^k\,. \tag{15.13}$$

Example 2. Taking account of relations (15.12) and (15.13), we find by (15.10) that

$$AF^k = a_{i_1\dots i_k} A(e^{i_1} \otimes \cdots \otimes e^{i_k}) ,$$

so that it is of interest to find $A(e^{i_1} \otimes \cdots \otimes e^{i_k})$.

From Definition (15.11), taking account of the relation $e^i(x) = x^i$, we find

$$A(e^{j_1}\otimes \cdots \otimes e^{j_k})(x_1, \dots, x_k) = \frac{1}{k!} e^{j_1}(x_{i_1}) \cdot \ldots \cdot e^{j_k}(x_{i_k}) \delta^{i_1\dots i_k}_{1\dots k} =$$

$$= \frac{1}{k!} x^{j_1}_{i_1} \cdot \ldots \cdot x^{j_k}_{i_k} \delta^{i_1\dots i_k}_{1\dots k} = \frac{1}{k!} \begin{vmatrix} x_1^{j_1} & \cdots & x_1^{j_k} \\ \cdots & \cdots & \cdots \\ x_k^{j_1} & \cdots & x_k^{j_k} \end{vmatrix} . \tag{15.14}$$

The tensor product of skew-symmetric forms is in general not skew-symmetric, so that we introduce the following *exterior product* in the class of skew-symmetric forms:

$$\omega^k \wedge \omega^l := \frac{(k+l)!}{k!l!} A(\omega^k \otimes \omega^l) . \tag{15.15}$$

Thus $\omega^k \wedge \omega^l$ is a skew-symmetric form ω^{k+l} of degree $k + l$.

Example 3. Based on the result (15.14) of Example 2, we find by Definition (15.15) that

$$e^{i_1} \wedge e^{i_2}(x_1, x_2) = \frac{2!}{1!1!} A(e^{i_1} \otimes e^{i_2})(x_1, x_2) =$$

$$= \begin{vmatrix} e^{i_1}(x_1) & e^{i_2}(x_1) \\ e^{i_1}(x_2) & e^{i_2}(x_2) \end{vmatrix} = \begin{vmatrix} x_1^{i_1} & x_1^{i_2} \\ x_2^{i_1} & x_2^{i_2} \end{vmatrix} . \tag{15.16}$$

Example 4. Using the equality obtained in Example 3, relation (15.14), and the definitions (15.11) and (15.15), we can write

$$e^{i_1} \wedge (e^{i_2} \wedge e^{i_3})(x_1, x_2, x_3) =$$

$$= \frac{(1+2)!}{1!2!} A\big(e^{i_1} \otimes (e^{i_2} \otimes e^{i_3})\big)(x_1, x_2, x_3) =$$

$$= \frac{3!}{1!2!} e^{i_1}(x_{j_1})(e^{i_2} \wedge e^{i_3})(x_{j_2}, x_{j_3}) \delta^{j_1 j_2 j_3}_{1\,2\,3} = \frac{1}{2!} x^{i_1}_{j_1} \begin{vmatrix} x^{i_2}_{j_2} & x^{i_3}_{j_2} \\ x^{i_2}_{j_3} & x^{i_3}_{j_3} \end{vmatrix} \delta^{j_1 j_2 j_3}_{1\,2\,3} =$$

$$= x_1^{i_1} \begin{vmatrix} x_2^{i_2} & x_2^{i_3} \\ x_3^{i_2} & x_3^{i_3} \end{vmatrix} - x_2^{i_1} \begin{vmatrix} x_1^{i_2} & x_1^{i_3} \\ x_3^{i_2} & x_3^{i_3} \end{vmatrix} + x_3^{i_1} \begin{vmatrix} x_1^{i_2} & x_1^{i_3} \\ x_2^{i_2} & x_2^{i_3} \end{vmatrix} =$$

$$= \begin{vmatrix} x_1^{i_1} & x_1^{i_2} & x_1^{i_3} \\ x_2^{i_1} & x_2^{i_2} & x_2^{i_3} \\ x_3^{i_1} & x_3^{i_2} & x_3^{i_3} \end{vmatrix} .$$

A similar computation shows that

$$e^{i_1} \wedge (e^{i_2} \wedge e^{i_3}) = (e^{i_1} \wedge e^{i_2}) \wedge e^{i_3} . \tag{15.17}$$

Using the expansion of the determinant along a column, we conclude by induction that

$$e^{i_1} \wedge \cdots \wedge e^{i_k}(x_1, \dots, x_k) = \begin{vmatrix} e^{i_1}(x_1) & \cdots & e^{i_k}(x_1) \\ \dots & \dots & \dots \\ e^{i_1}(x_k) & \cdots & e^{i_k}(x_k) \end{vmatrix}, \tag{15.18}$$

and, as one can see from the computations just carried out, formula (15.18) holds for any 1-forms $e^{i_1}, \dots, e^{i_k}$ (not just the basis forms of the space X^*).

Taking the properties of the tensor product and the alternation operation listed above into account, we obtain the following properties of the exterior product of skew-symmetric forms:

$$(\omega_1^k + \omega_2^k) \wedge \omega^l = \omega_1^k \wedge \omega^l + \omega_2^k \wedge \omega^l, \tag{15.19}$$

$$(\lambda\omega^k) \wedge \omega^l = \lambda(\omega^k \wedge \omega^l), \tag{15.20}$$

$$\omega^k \wedge \omega^l = (-1)^{kl}\omega^l \wedge \omega^k, \tag{15.21}$$

$$(\omega^k \wedge \omega^l) \wedge \omega^m = \omega^k \wedge (\omega^l \wedge \omega^m). \tag{15.22}$$

Proof. Equalities (15.19) and (15.20) follow obviously from relations (15.6)–(15.8) and (15.12) and (15.13).

From relations (15.10)–(15.14) and (15.17), for every skew-symmetric form $\omega = a_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}$ we obtain

$$\omega = A\omega = a_{i_1 \dots i_k} A(e^{i_1} \otimes \cdots \otimes e^{i_k}) = \frac{1}{k!} a_{i_1 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

Using the equalities (15.19) and (15.20) we see that it now suffices to verify (15.21) and (15.22) for the forms $e^{i_1} \wedge \cdots \wedge e^{i_k}$.

Associativity (15.22) for such forms was already established by (15.17).

We now obtain (15.21) immediately from (15.18) and the properties of determinants for these particular forms. □

Along the way we have shown that every form $\omega \in \Omega^k$ can be represented as

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}. \tag{15.23}$$

Thus, the set $\Omega = \{\Omega^k\}$ of skew-symmetric forms on the vector space X relative to the linear vector-space operations (15.3) and (15.4) and the exterior multiplication (15.15) is a graded algebra $\Omega = \bigoplus_{k=0}^{\dim X} \Omega^k$. The vector-space operations on Ω are carried out inside each vector space Ω^k, and if $\omega^k \in \Omega^k$, $\omega^l \in \Omega^l$, then $\omega^k \wedge \omega^l \in \Omega^{k+l}$.

In the direct sum $\bigoplus \Omega^k$ the summation runs from zero the dimension of the space X, since the skew-symmetric forms $\omega^k : X^k \to \mathbb{R}$ of degree larger than the dimension of X are necessarily identically zero, as one can see by (15.21) (or from relations (15.23) and (15.8)).

15.1.3 Linear Mappings of Vector Spaces and the Adjoint Mappings of the Conjugate Spaces

Let X and Y be vector spaces over the field $\mathbb{R}$ of real numbers (or any other field, so long as it is the same field for both X and Y), and let $l : X \to Y$ be a linear mapping of X into Y, that is, for every $x, x_1, x_2 \in X$ and every $\lambda \in \mathbb{R}$,

$$l(x_1 + x_2) = l(x_1) + l(x_2) \text{ and } l(\lambda x) = \lambda l(x) . \tag{15.24}$$

A linear mapping $l : X \to Y$ naturally generates its adjoint mapping $l^* : \mathcal{F}_Y \to \mathcal{F}_X$ from the set of linear functionals on Y ($\mathcal{F}_Y$) into the analogous set $\mathcal{F}_X$. If F_Y^k is a k-form on Y, then by definition

$$\left(l^* F_Y^k\right)(x_1, \dots, x_k) := F_Y^k(lx_1, \dots, lx_k) . \tag{15.25}$$

It can be seen by (15.24) and (15.25) that $l^* F_Y^k$ is a k-form F_X^k on X, that is, $l^*(\mathcal{F}_Y^k) \subset \mathcal{F}_X^k$. Moreover, if the form F_Y^k was skew-symmetric, then $(l^* F_Y^k) = F_X^k$ is also skew-symmetric, that is, $l^*(\Omega_Y^k) \subset \Omega_X^k$. Inside each vector space $\mathcal{F}_Y^k$ and Ω_Y^k the mapping l^* is obviously linear, that is,

$$l^*(F_1^k + F_2^k) = l^* F_1^k + l^* F_2^k \quad \text{and} \quad l^*(\lambda F^k) = \lambda l^* F^k . \tag{15.26}$$

Now comparing definition (15.25) with the definitions (15.5), (15.11), and (15.15) of the tensor product, alternation, and exterior product of forms, we conclude that

$$l^*(F^p \otimes F^q) = (l^* F^p) \otimes (l^* F^q) , \tag{15.27}$$

$$l^*(AF^p) = A(l^* F^p) , \tag{15.28}$$

$$l^*(\omega^p \wedge \omega^q) = (l^* \omega^p) \wedge (l^* \omega^q) . \tag{15.29}$$

Example 5. Let $e_1, \dots, e_m$ be a basis in X, $\widetilde{e}_1, \dots, \widetilde{e}_n$ a basis in Y, and $l(e_i) = c_i^j \widetilde{e}_j$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$. If the k-form F_Y^k has the coordinate representation

$$F_Y^k(y_1, \dots, y_k) = b_{j_1 \dots j_k} y_1^{j_1} \cdot \ldots \cdot y_k^{j_k}$$

in the basis $\widetilde{e}_1, \dots, \widetilde{e}_n$, where $b_{j_1 \dots j_k} = F_Y^k(\widetilde{e}_{j_1}, \dots, \widetilde{e}_{j_k})$, then

$$(l^* F_Y^k)(x_1, \dots, x_k) = a_{i_1 \dots i_k} x_1^{i_1} \cdot \ldots \cdot x_k^{i_k} ,$$

where $a_{i_1 \dots i_k} = b_{j_1 \dots j_k} c_{i_1}^{j_1} \cdot \ldots \cdot c_{i_k}^{j_k}$, since

$$a_{i_1 \dots i_k} =: (l^* F_Y^k)(e_{i_1}, \dots, e_{i_k}) := F_Y^k(le_{i_1}, \dots, le_{i_k}) =$$
$$= F_Y^k(c_{i_1}^{j_1} \widetilde{e}_{j_1}, \dots, c_{i_k}^{j_k} \widetilde{e}_{j_k}) = F_Y^k(\widetilde{e}_{j_1}, \dots, \widetilde{e}_{j_k}) c_{i_1}^{j_1} \cdot \ldots \cdot c_{i_k}^{j_k} .$$

Example 6. Let $e^1, \dots, e^m$ and $\widetilde{e}^1, \dots, \widetilde{e}^n$ be the bases of the conjugate spaces X^* and Y^* dual to the bases in Example 5. Under the hypotheses of Example 5 we obtain

$$(l^* \widetilde{e}^j)(x) = (l^* \widetilde{e}^j)(x^i e_i) = \widetilde{e}^j(x^i l e_i) = x^i \widetilde{e}^j(c_i^k \widetilde{e}_k) =$$
$$= x^i c_i^k \widetilde{e}^j(\widetilde{e}_k) = x^i c_i^k \delta_k^j = c_i^j x^i = c_i^j e^i(x) .$$

Example 7. Retaining the notation of Example 6 and taking account of relations (15.22) and (15.29), we now obtain

$$l^*(\widetilde{e}^{j_1} \wedge \cdots \wedge \widetilde{e}^{j_k}) = l^*\widetilde{e}^{j_1} \wedge \cdots \wedge \widetilde{l}^* e^{j_k} =$$
$$(c^{j_1}_{i_1} e^{i_1}) \wedge \cdots \wedge (c^{j_k}_{i_k} e^{i_k}) = c^{j_1}_{i_1} \cdot \ldots \cdot c^{j_k}_{i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} =$$
$$= \sum_{1 \le i_1 < \cdots < i_k \le m} \begin{vmatrix} c^{j_1}_{i_1} & \cdots & c^{j_k}_{i_1} \\ \cdots & \cdots & \cdots \\ c^{j_1}_{i_k} & \cdots & c^{j_k}_{i_k} \end{vmatrix} e^{i_1} \wedge \cdots \wedge e^{i_k} .$$

Keeping Eq. (15.26) in mind, we can conclude from this that

$$l^*\Bigl(\sum_{1 \le j_1 < \cdots < j_k \le n} b_{j_1 \ldots j_k} \widetilde{e}^{j_1} \wedge \cdots \wedge \widetilde{e}^{j_k} \Bigr) =$$
$$= \sum_{\substack{1 \le i_1 < \cdots < i_k \le m \\ 1 \le j_1 < \cdots < j_k \le n}} b_{j_1 \ldots j_k} \begin{vmatrix} c^{j_1}_{i_1} & \cdots & c^{j_k}_{i_1} \\ \cdots & \cdots & \cdots \\ c^{j_1}_{i_k} & \cdots & c^{j_k}_{i_k} \end{vmatrix} e^{i_1} \wedge \cdots \wedge e^{i_k} =$$
$$= \sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1 \ldots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} .$$

15.1.4 Problems and Exercises

1. Show by examples that in general

a) $F^k \otimes F^l \neq F^l \otimes F^k$;

b) $A(F^k \otimes F^l) \neq AF^k \otimes AF^l$;

c) if $F^k, F^l \in \Omega$, then it is not always true that $F^k \otimes F^l \in \Omega$.

2. a) Show that if $e_1, \ldots, e_n$ is a basis of the vector space X and the linear functionals $e^1, \ldots, e^n$ on X (that is elements of the conjugate space X^*) are such that $e^j(e_i) = \delta^j_i$, then $e^1, \ldots, e^n$ is a basis in X^*.

b) Verify that one can always form a basis of the space $\mathcal{F}^k = \mathcal{F}^k(X)$ from k-forms of the form $e^{i_1} \otimes \cdots \otimes e^{i_k}$, and find the dimension ($\dim \mathcal{F}^k$) of this space, knowing that $\dim X = n$.

c) Verify that one can always form a basis of the space Ω^k from forms of the form $e^{i_1} \wedge \cdots \wedge e^{i_k}$, and find $\dim \Omega^k$ knowing that $\dim X = n$.

d) Show that if $\Omega = \bigoplus_{k=0}^{k=n} \Omega^k$, then $\dim \Omega = 2^n$.

3. The *exterior (Grassmann)*[1] *algebra* G over a vector space X and a field P (usually denoted $\bigwedge(X)$ in agreement with the symbol $\wedge$ for the multiplication operation in G) is defined as the associative algebra with identity 1 having the following properties:

1^0 G is generated by the identity and X, that is, any subalgebra of G containing 1 and X is equal to G;

2^0 $x \wedge x = 0$ for every vector $x \in X$;

3^0 $\dim G = 2^{\dim X}$.

a) Show that if $e_1, \dots, e_n$ is a basis in X, then the set $1, e_1, \dots, e_n, e_1 \wedge e_2, \dots, e_{n-1} \wedge e_n, \dots, e_1 \wedge \dots \wedge e_n$ of elements of G of the form $e_{i_1} \wedge \dots \wedge e_{i_k} = e_I$, where $I = \{i_1 < \dots < i_k\} \subset \{1, 2, \dots, n\}$, forms a basis in G.

b) Starting from the result in a) one can carry out the following formal construction of the algebra $G = \bigwedge(X)$.

For the subsets $I = \{i_1, \dots, i_k\}$ of $\{1, 2, \dots, n\}$ shown in a) we form the formal elements e_I, (by identifying $e_{\{i\}}$ with e_i, and $e_\varnothing$ with 1), which we take as a basis of the vector space G over the field P. We define multiplication in G by the formula

$$\left(\sum_I a_I e_I\right)\left(\sum_J b_J e_J\right) = \sum_{I,J} a_I b_J \varepsilon(I, J) e_{I \cup J} \,,$$

where $\varepsilon(I, J) = \operatorname{sgn} \prod\limits_{i \in I, j \in J} (j - i)$. Verify that the Grassmann algebra $\bigwedge(X)$ is obtained in this way.

c) Prove the uniqueness (up to isomorphism) of the algebra $\bigwedge(X)$.

d) Show that the algebra $\bigwedge(X)$ is graded: $\bigwedge(X) = \bigoplus\limits_{k=0}^{k=n} \bigwedge^k(X)$, where $\bigwedge^k(X)$ is the linear span of the elements of the form $e_{i_1} \wedge \dots \wedge e_{i_k}$; here if $a \in \bigwedge^p(X)$ and $b \in \bigwedge^q(X)$, then $a \wedge b \in \bigwedge^{p+q}(X)$. Verify that $a \wedge b = (-1)^{pq} b \wedge a$.

4. a) Let $A : X \to Y$ be a linear mapping of X into Y. Show that there exists a unique homomorphism $\bigwedge(A) : \bigwedge(X) \to \bigwedge(Y)$ from $\bigwedge(X)$ into $\bigwedge(Y)$ that agrees with A on the subspace $\bigwedge'(X) \subset \bigwedge(X)$ identified with X.

b) Show that the homomorphism $\bigwedge(A)$ maps $\bigwedge^k(X)$ into $\bigwedge^k(Y)$. The restriction of $\bigwedge(A)$ to $\bigwedge^k(X)$ is denoted by $\bigwedge^k(A)$.

c) Let $\{e_i : i = 1, \dots, m\}$ be a basis in X and $\{e_j : j = 1, \dots, n\}$ a basis in Y, and let the matrix (a_j^i) correspond to the operator A in these bases. Show that if $\{e_I : I \subset \{1, \dots, m\}\}$, $\{e_J : J \subset \{1, \dots, n\}\}$ are the corresponding bases of the spaces $\bigwedge(X)$ and $\bigwedge(Y)$, then the matrix of the operator $\bigwedge^k(A)$ has the form $a_J^I = \det(a_j^i)$, $i \in I$, $j \in J$, where $\operatorname{card} I = \operatorname{card} J = k$.

d) Verify that if $A : X \to Y$, $B : Y \to Z$ are linear operators, then the equality $\bigwedge(B \circ A) = \bigwedge(B) \circ \bigwedge(A)$ holds.

[1] H. Grassmann (1809–1877) – German mathematician, physicist and philologist; in particular, he created the first systematic theory of multidimensional and Euclidean vector spaces and gave the definition of the inner product of vectors.

15.2 Manifolds

15.2.1 Definition of a Manifold

Definition 1. A Hausdorff topological space whose topology has a countable base[2] is called an *n-dimensional manifold* if each of its points has a neighborhood U homeomorphic either to all of $\mathbb{R}^n$ or to the half-space $H^n = \{x \in \mathbb{R}^n \big| x^1 \le 0\}$.

Definition 2. A mapping $\varphi : \mathbb{R}^n \to U \subset M$ (or $\varphi : H^n \to U \subset M$) that realizes the homeomorphism of Definition 1 is a *local chart of the manifold* M, $\mathbb{R}^n$ (or H^n) is called the *parameter domain*, and U the *range* of the chart on the manifold M.

A local chart endows each point $x \in U$ with the coordinates of the point $t = \varphi^{-1}(x) \in \mathbb{R}^n$ corresponding to it. Thus, a local coordinate system is introduced in the region U; for that reason the mapping φ, or, in more expanded notation, the pair (U, φ) is a map of the region U in the ordinary meaning of the term.

Definition 3. A set of charts whose ranges taken together cover the entire manifold is called an *atlas* of the manifold.

Example 1. The sphere $S^2 = \{x \in \mathbb{R}^3 \big| |x| = 1\}$ is a two-dimensional manifold. If we interpret S^2 as the surface of the Earth, then an atlas of geographical maps will be an atlas of the manifold S^2.

The one-dimensional sphere $S^1 = \{x \in \mathbb{R}^2 \big| |x| = 1\}$ – a circle in $\mathbb{R}^2$ – is obviously a one-dimensional manifold. In general, the sphere $S^n = \{x \in \mathbb{R}^{n+1} \big| |x| = 1\}$ is an n-dimensional manifold. (See Sect. 12.1.)

Remark 1. The object (the manifold M) introduced by Definition 1 obviously does not change if we replace $\mathbb{R}^n$ and H^n by any parameter domains in $\mathbb{R}^n$ homeomorphic to them. For example, such a domain might be the open cube $I^n = \{x \in \mathbb{R}^n \big| 0 < x^i < 1, i = 1, \dots, n\}$ and the cube with a face attached $\tilde{I}^n = \{x \in \mathbb{R}^n \big| 0 < x^1 \le 1, 0 < x^i < 1, i = 2, \dots, n\}$. Such standard parameter domains are used quite often.

It is also not difficult to verify that the object introduced by Definition 1 does not change if we require only that each point $x \in M$ have a neighborhood U in M homeomorphic to some open subset of the half-space H^n.

Example 2. If X is an m-dimensional manifold with an atlas of charts $\{(U_\alpha, \varphi_\alpha)\}$ and Y is an n-dimensional manifold with atlas $\{(V_\beta, \psi_\beta)\}$, then $X \times Y$ can be regarded as an $(m+n)$-dimensional manifold with the atlas $\{(W_{\alpha\beta}, \chi_{\alpha\beta})\}$, where $W_{\alpha\beta} = U_\alpha \times V_\beta$ and the mapping $\chi_{\alpha\beta} = (\varphi_\alpha, \psi_\beta)$ maps the direct product of the domains of definition of φ_α and ψ_β into $W_{\alpha\beta}$.

[2] See Sect. 9.2 and also Remarks 2 and 3 in the present section.

In particular, the two-dimensional torus $T^2 = S^1 \times S^1$ (Fig. 12.1) or the n-dimensional torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ factors}}$ is a manifold of the corresponding dimension.

If the ranges U_i and U_j of two charts (U_i, φ_i) and (U_j, φ_j) of a manifold M intersect, that is, $U_i \cap U_j \neq \varnothing$, mutually inverse homeomorphisms $\varphi_{ij} : I_{ij} \to I_{ji}$ and $\varphi_{ji} : I_{ji} \to I_{ij}$ naturally arise between the sets $I_{ij} = \varphi_i^{-1}(U_j)$ and $I_{ji} = \varphi_j^{-1}(U_i)$. These homeomorphisms are given by $\varphi_{ij} = \varphi_j^{-1} \circ \varphi_i\big|_{I_{ij}}$ and $\varphi_{ji} = \varphi_i^{-1} \circ \varphi_j\big|_{I_{ji}}$. These homeomorphisms are often called *changes of coordinates*, since they effect a transition from one local coordinate system to another system of the same kind in their common range $U_i \cap U_j$ (Fig. 15.1).

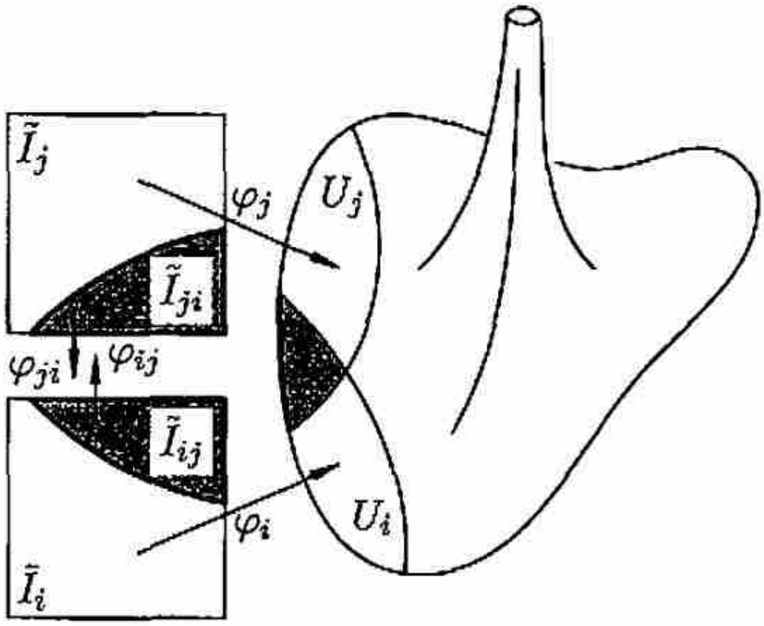

Fig. 15.1.

Definition 4. The number n in Definition 1 is the *dimension of the manifold* M and is usually denoted $\dim M$.

Definition 5. If a point $\varphi^{-1}(x)$ on the boundary ∂H^n of the half-space H^n corresponds to a point $x \in U$ under the homeomorphism $\varphi : H^n \to U$, then x is called a *boundary point of the manifold* M (and of the neighborhood U). The set of all boundary points of a manifold M is called the *boundary* of this manifold and is usually denoted ∂M.

By the topological invariance of interior points (Brouwer's theorem[3]) the concepts of dimension and boundary point of a manifold are unambiguously

[3] This theorem asserts that under a homeomorphism $\varphi : E \to \varphi(E)$ of a set $E \subset \mathbb{R}^n$ onto a set $\varphi(E) \subset \mathbb{R}^n$ the interior points of E map to interior points of $\varphi(E)$.

defined, that is, independent of the particular local charts used in Definitions 4 and 5. We have not proved Brouwer's theorem, but the invariance of interior points under diffeomorphisms is well-known to us (a consequence of the inverse function theorem). Since it is diffeomorphisms that we shall be dealing with, we shall not digress here to discuss Brouwer's theorem.

Example 3. The closed ball $\overline{B}^n = \{x \in \mathbb{R}^n \big| |x| \leq 1\}$ or, as we say, the *n-dimensional disk*, is an n-dimensional manifold whose boundary is the $(n-1)$-dimensional sphere $S^{n-1} = \{x \in \mathbb{R}^n \big| |x| = 1\}$.

Remark 2. A manifold M having a non-empty set of boundary points is usually called a *manifold with boundary*, the term *manifold* (in the proper sense of the term) being reserved for manifolds without boundary. In Definition 1 these cases are not distinguished.

Proposition 1. *The boundary ∂M of an n-dimensional manifold with boundary M is an $(n-1)$-dimensional manifold without boundary.*

Proof. Indeed, $\partial H^n = \mathbb{R}^{n-1}$, and the restriction to ∂H^n of a chart of the form $\varphi_i : H^n \to U_i$ belonging to an atlas of M generates an atlas of ∂M. □

Example 4. Consider the planar double pendulum (Fig. 15.2) with arm a shorter than arm b, both being free to oscillate, except that the oscillations of b are limited in range by barriers. The configuration of such a system is characterized at each instant of time by the two angles α and β. If there were no constraints, the configuration space of the double pendulum could be identified with the two-dimensional torus $T^2 = S^1_\alpha \times S^1_\beta$.

Under these constraints, the configuration space of the double pendulum is parametrized by the points of the cylinder $S^1_\alpha \times I^1_\beta$, where S^1_α is the circle, corresponding to all possible positions of the arm a, and $I^1_\beta = \{\beta \in \mathbb{R} \big| |\beta| \leq$

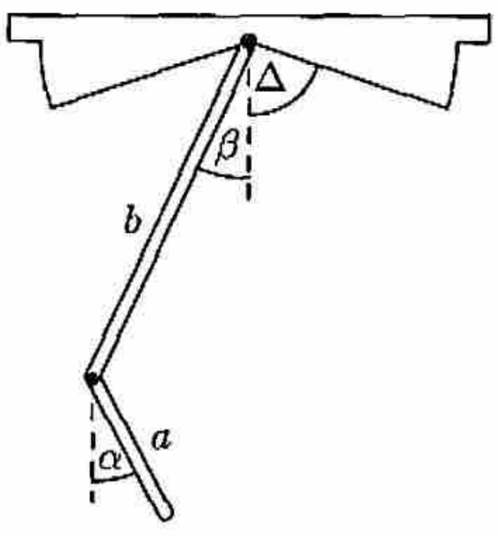

Fig. 15.2.

$\Delta\}$ is the interval within which the angle β may vary, characterizing the position of the arm b.

In this case we obtain a manifold with boundary. The boundary of this manifold consists of the two circles $S^1_\alpha \times \{-\Delta\}$ and $S^1_\alpha \times \{\Delta\}$, which are the products of the circle S^1_α and the endpoints $\{-\Delta\}$ and $\{\Delta\}$ of the interval I^1_β.

Remark 3. It can be seen from Example 4 just considered that coordinates sometimes arise naturally on M (α and β in this example), and they themselves induce a topology on M. Hence, in Definition 1 of a manifold, it is not always necessary to require in advance that M have a topology. The essence of the concept of a manifold is that the points of some set M can be parametrized by the points of a set of subdomains of $\mathbb{R}^n$. A natural connection then arises between the coordinate systems that thereby arise on parts of M, expressed in the mappings of the corresponding domains of $\mathbb{R}^n$. Hence we can assume that M is obtained from a collection of domains of $\mathbb{R}^n$ by exhibiting some rule for identifying their points or, figuratively speaking, exhibiting a rule for gluing them together. Thus defining a manifold essentially means giving a set of subdomains of $\mathbb{R}^n$ and a rule of correspondence for the points of these subdomains. We shall not take the time to make this any more precise by formalizing the concept of gluing or identifying points, introducing a topology on M, and the like.

Definition 6. A manifold is *compact* (resp. *connected*) if it is compact (resp. connected) as a topological space.

The manifolds considered in Examples 1–4 are compact and connected. The boundary of the cylinder $S^1_\alpha \times I^1_\beta$ in Example 4 consists of two independent circles and is a one-dimensional compact, but not connected, manifold. The boundary $S^{n-1} = \partial\overline{B}^n$ of the n-dimensional disk of Example 3 is a compact manifold, which is connected for $n > 1$ and disconnected (it consists of two points) if $n = 1$.

Example 5. The space $\mathbb{R}^n$ itself is obviously a connected noncompact manifold without boundary, and the half-space H^n provides the simplest example of a connected noncompact manifold with boundary. (In both cases the atlas can be taken to consist of the single chart corresponding to the identity mapping.)

Proposition 2. *If a manifold M is connected, it is path connected.*

Proof. After fixing a point $x_0 \in M$, consider the set E_{x_0} of points of M that can be joined to x_0 by a path in M. The set E_{x_0}, as one can easily verify from the definition of a manifold, is both open and closed in M. But that means that $E_{x_0} = M$. $\square$

Example 6. If to each real $n \times n$ matrix we assign the point of $\mathbb{R}^{n^2}$ whose coordinates are obtained by writing out the elements of the matrix in some fixed order, then the group $GL(n, \mathbb{R})$ of nonsingular $n \times n$ matrices becomes a manifold of dimension n^2. This manifold is noncompact (the elements of the matrices are not bounded) and nonconnected. This last fact follows from the fact that $GL(n, \mathbb{R})$ contains matrices with both positive and negative determinants. The points of $GL(n, \mathbb{R})$ corresponding to two such matrices cannot be joined by a path. (On such a path there would have to be a point corresponding to a matrix whose determinant is zero.)

Example 7. The group $SO(2, \mathbb{R})$ of orthogonal mappings of the plane $\mathbb{R}^2$ having determinant equal to 1 consists of matrices of the form $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ and hence can be regarded as a manifold that is identified with the circle – the domain of variation of the angular parameter α. Thus $SO(2, \mathbb{R})$ is a one-dimensional compact connected manifold. If we also allow reflections about lines in the plane $\mathbb{R}^2$, we obtain the group $O(2, \mathbb{R})$ of all real orthogonal 2×2 matrices. It can be naturally identified with two different circles, corresponding to matrices with determinants $+1$ and -1 respectively. That is, $O(2, \mathbb{R})$ is a one-dimensional compact, but not connected manifold.

Example 8. Let $\mathbf{a}$ be a vector in $\mathbb{R}^2$ and $T_{\mathbf{a}}$ the group of rigid motions of the plane generated by $\mathbf{a}$. The elements of $T_{\mathbf{a}}$ are translations by vectors of the form $n\mathbf{a}$, where $n \in \mathbb{Z}$. Under the action of the elements g of the group $T_{\mathbf{a}}$ each point x of the plane is displaced to a point $g(x)$ of the form $x + n\mathbf{a}$. The set of all points to which a given point $x \in \mathbb{R}^2$ passes under the action of the elements of this group of transformations is called its *orbit*. The property of points of $\mathbb{R}^2$ of belonging to the same orbit is obviously an equivalence relation on $\mathbb{R}^2$, and the orbits are the equivalence classes of this relation. A domain in $\mathbb{R}^2$ containing one point from each equivalence class is called a *fundamental domain* of this group of automorphisms (for a more precise statement see Problem 5d)).

In the present case we can take as a fundamental domain a strip of width $|\mathbf{a}|$ bounded by two parallel lines orthogonal to $\mathbf{a}$. We need only take into account that these lines themselves are obtained from each other through translations by $\mathbf{a}$ and $-\mathbf{a}$ respectively. Inside a strip of width less than $|\mathbf{a}|$ and orthogonal to $\mathbf{a}$ there are no equivalent points, so that all orbits having representatives in that strip are endowed uniquely with the coordinates of their representatives. Thus the quotient set $\mathbb{R}^2/T_{\mathbf{a}}$ consisting of orbits of the group $T_{\mathbf{a}}$ becomes a manifold. From what was said above about a fundamental domain, one can easily see that this manifold is homeomorphic to the cylinder obtained by gluing the boundary lines of a strip of width $|\mathbf{a}|$ together at equivalent points.

Example 9. Now let $\mathbf{a}$ and $\mathbf{b}$ be a pair of orthogonal vectors of the plane $\mathbb{R}^2$ and $T_{\mathbf{a},\mathbf{b}}$ the group of translations generated by these vectors. In this case a fundamental domain is the rectangle with sides $\mathbf{a}$ and $\mathbf{b}$. Inside this rectangle the only equivalent points are those that lie on opposite sides. After gluing the sides of this fundamental rectangle together, we verify that the resulting manifold $\mathbb{R}^2/T_{\mathbf{a},\mathbf{b}}$ is homeomorphic to the two-dimensional torus.

Example 10. Now consider the group $G_{a,b}$ of rigid motions of the plane $\mathbb{R}^2$ generated by the transformations $a(x,y)=(x+1,1-y)$ and $b(x,y)=(x,y+1)$.

A fundamental domain for the group $G_{a,b}$ is the unit square whose horizontal sides are identified at points lying on the same vertical line, but whose vertical sides are identified at points symmetric about the center. Thus the resulting manifold $\mathbb{R}^2/G_{a,b}$ turns out to be homeomorphic to the Klein bottle (see Sect. 12.1).

We shall not take time to discuss here the useful and important examples studied in Sect. 12.1.

15.2.2 Smooth Manifolds and Smooth Mappings

Definition 7. An atlas of a manifold is *smooth (of class $C^{(k)}$ or analytic)* if all the coordinate-changing functions for the atlas are smooth mappings (diffeomorphisms) of the corresponding smoothness class.

Two atlases of a given smoothness (the same smoothness for both) are *equivalent* if their union is an atlas of this smoothness.

Example 11. An atlas consisting of a single chart can be regarded as having any desired smoothness. Consider in this connection the atlas on the line $\mathbb{R}^1$ generated by the identity mapping $\mathbb{R}^1 \ni x \mapsto \varphi(x) = x \in \mathbb{R}^1$, and a second atlas – generated by any strictly monotonic function $\mathbb{R}^1 \ni x \mapsto \widetilde{\varphi}(x) \in \mathbb{R}^1$, mapping $\mathbb{R}^1$ onto $\mathbb{R}^1$. The union of these atlases is an atlas having smoothness equal to the smaller of the smoothnesses of $\widetilde{\varphi}$ and $\widetilde{\varphi}^{-1}$.

In particular, if $\widetilde{\varphi}(x) = x^3$, then the atlas consisting of the two charts $\{x, x^3\}$ is not smooth, since $\widetilde{\varphi}^{-1}(x) = x^{1/3}$. Using what has just been said, we can construct infinitely smooth atlases in $\mathbb{R}^1$ whose union is an atlas of a preassigned smoothness class $C^{(k)}$.

Definition 8. A *smooth manifold (of class $C^{(k)}$ or analytic)* is a manifold M with an equivalence class of atlases of the given smoothness.

After this definition the following terminology is comprehensible: *topological manifold* (of class $C^{(0)}$), *$C^{(k)}$-manifold*, *analytic manifold*.

To give the entire equivalence class of atlases of a given smoothness on a manifold M it suffices to give any atlas A of this equivalence class. Thus we can assume that a smooth manifold is a pair (M, A), where M is a manifold and A an atlas of the given smoothness on M.

The set of equivalent atlases of a given smoothness on a manifold is often called a *structure of this smoothness on the manifold.* There may be different smooth structures of even the same smoothness on a given topological manifold (see Example 11 and Problem 3).

Let us consider some more examples in which our main attention is directed to the smoothness of the coordinate changes.

Example 12. The one-dimensional manifold $\mathbb{RP}^1$ called the *real projective line*, is the pencil of lines in $\mathbb{R}^2$ passing through the origin, with the natural notion of distance between two lines (measured, for example, by the magnitude of the smaller angle between them). Each line of the pencil is uniquely determined by a nonzero direction vector (x^1, x^2), and two such vectors give the same line if and only if they are collinear. Hence $\mathbb{RP}^1$ can be regarded as a set of equivalence classes of ordered pairs (x^1, x^2) of real numbers. Here at least one of the numbers in the pair must be nonzero, and two pairs are considered equivalent (identified) if they are proportional. The pairs (x^1, x^2) are usually called *homogeneous coordinates* on $\mathbb{RP}^1$. Using the interpretation of $\mathbb{RP}^1$ in homogeneous coordinates, it is easy to construct an atlas of two charts on $\mathbb{RP}^1$. Let U_i, $i = 1, 2$, be the lines (classes of pairs (x^1, x^2)) in $\mathbb{RP}^1$ for which $x^i \neq 0$. To each point (line) $p \in U_1$ there corresponds a unique pair $\left(1, \frac{x^2}{x^1}\right)$ determined by the number $t_1^2 = \frac{x^2}{x^1}$. Similarly the points of the region U_2 are in one-to-one correspondence with pairs of the form $\left(\frac{x^1}{x^2}, 1\right)$ and are determined by the number $t_2^1 = \frac{x^1}{x^2}$. Thus local coordinates arise in U_1 and U_2, which obviously correspond to the topology introduced above on $\mathbb{RP}^1$. In the common range $U_1 \cap U_2$ of these local charts the coordinates they introduce are connected by the relations $t_2^1 = (t_1^2)^{-1}$ and $t_1^2 = (t_2^1)^{-1}$, which shows that the atlas is not only $C^{(\infty)}$ but even analytic.

It is useful to keep in mind the following interpretation of the manifold $\mathbb{RP}^1$. Each line of the original pencil of lines is completely determined by its intersection with the unit circle. But there are exactly two such points, diametrically opposite to each other. Lines are near if and only if the corresponding points of the circle are near. Hence $\mathbb{RP}^1$ can be interpreted as a circle with diametrically opposite points identified (glued together). If we take only a semicircle, there is only one pair of identified points on it, the end-points. Gluing them together, we again obtain a topological circle. Thus $\mathbb{RP}^1$ is homeomorphic to the circle as a topological space.

Example 13. If we now consider the pencil of lines passing through the origin in $\mathbb{R}^3$, or, what is the same, the set of equivalence classes of ordered triples of points (x^1, x^2, x^3) of real numbers that are not all three zero, we obtain the *real projective plane* $\mathbb{RP}^2$. In the regions U_1, U_2, and U_3 where $x^1 \neq 0$, $x^2 \neq 0$, $x^3 \neq 0$ respectively, we introduce local coordinate systems $\left(1, \frac{x^2}{x^1}, \frac{x^3}{x^1}\right) = (1, t_1^2, t_1^3) \sim (t_1^2, t_1^3)$, $\left(\frac{x^1}{x^2}, 1, \frac{x^3}{x^2}\right) = (t_2^1, 1, t_2^3) \sim (t_2^1, t_2^3)$, and $\left(\frac{x^1}{x^3}, \frac{x^2}{x^1}, 1\right) = (t_3^1, t_3^2, 1) \sim (t_3^1, t_3^2)$, which are obviously connected by the relations $t_i^j = (t_j^i)^{-1}$, $t_i^j = t_k^j(t_k^i)^{-1}$, which apply in the common portions of the ranges of these charts.

For example, the transition from (t_1^2, t_1^3) to (t_2^1, t_2^3) in the domain $U_1 \cap U_2$ is given by the formulas

$$t_2^1 = (t_1^2)^{-1}, \quad t_2^3 = t_1^3 \cdot (t_1^2)^{-1}.$$

The Jacobian of this transformation is $-(t_1^2)^{-3}$, and since $t_1^2 = \frac{x^2}{x^1}$, it is defined and nonzero at points of the set $U_1 \cap U_2$ under consideration.

Thus $\mathbb{RP}^2$ is a two-dimensional manifold having an analytic atlas consisting of three charts.

By the same considerations as in Example 12, where we studied the projective line $\mathbb{RP}^1$, we can interpret the projective plane $\mathbb{RP}^2$ as the two-dimensional sphere $S^2 \subset \mathbb{R}^2$ with antipodal points identified, or as a hemisphere, with diametrically opposite points of its boundary circle identified. Projecting the hemisphere into the plane, we obtain the possibility of interpreting $\mathbb{RP}^2$ as a (two-dimensional) disk with diametrically opposite points of its boundary circle identified.

Example 14. The set of lines in the plane $\mathbb{R}^2$ can be parititioned into two sets: U, the nonvertical lines, and V, the nonhorizontal lines. Each line in U has an equation of the form $y = u_1 x + u_2$, and hence is characterized by the coordinates (u_1, u_2), while each line in V has an equation $x = v_1 y + v_2$ and is determined by coordinates (v_1, v_2). For lines in the intersection $U \cap V$ we have the coordinate transformation $v_1 = u_1^{-1}$, $v_2 = -u_2 u_1^{-1}$ and $u_1 = v_1^{-1}$, $u_2 = -v_2 v_1^{-1}$. Thus this set is endowed with an analytic atlas consisting of two charts.

Every line in the plane has an equation $ax+by+c = 0$ and is characterized by a triple of numbers (a, b, c), proportional triples defining the same line. For that reason, it might appear that we are again dealing with the projective plane $\mathbb{RP}^2$ considered in Example 13. However, whereas in $\mathbb{RP}^2$ we admitted any triples of numbers not all zero, now we do not admit triples of the form $(0, 0, c)$ where $c \neq 0$. A single point in $\mathbb{RP}^2$ corresponds to the set of all such triples. Hence the manifold obtained in our present example is homeomorphic to the one obtained from $\mathbb{RP}^2$ by removing one point. If we interpret $\mathbb{RP}^2$ as a disk with diametrically opposite points of the boundary circle identified, then, deleting the center of the circle, we obtain, up to homeomorphism, an annulus whose outer circle is glued together at diametrically opposite points. By a simple incision one can easily show that the result is none other than the familiar Möbius band.

Definition 9. Let M and N be $C^{(k)}$-manifolds. A mapping $f : M \to N$ is *l-smooth* (a $C^{(l)}$-mapping) if the local coordinates of the point $f(x) \in N$ are $C^{(l)}$-functions of the local coordinates of $x \in M$.

This definition has an unambiguous meaning (one that is independent of the choice of local coordinates) if $l \leq k$.

In particular, the smooth mappings of M into $\mathbb{R}^1$ are smooth functions on M, and the smooth mappings of $\mathbb{R}^1$ (or an interval of $\mathbb{R}^1$) into M are smooth paths on M.

Thus the degree of smoothness of a function $f : M \to N$ on a manifold M cannot exceed the degree of smoothness of the manifold itself.

15.2.3 Orientation of a Manifold and its Boundary

Definition 10. Two charts of a smooth manifold are *consistent* if the transition from the local coordinates in one to the other in their common range is a diffeomorphism whose Jacobian is everywhere positive.

In particular, if the ranges of two local charts have empty intersection, they are considered consistent.

Definition 11. An atlas A of a smooth manifold (M, A) is an *orienting atlas of M* if it consists of pairwise consistent charts.

Definition 12. A manifold is *orientable* if it has an orienting atlas. Otherwise it is *nonorientable*.

Two orienting atlases of a manifold will be regarded as *equivalent* (in the sense of the question of orientation of the manifold considered just now) if their union is also an orienting atlas of the manifold. It is easy to see that this relation really is an equivalence relation.

Definition 13. An equivalence class of orienting atlases of a manifold in the relation just defined is called an *orientation class of atlases of the manifold* or an *orientation of the manifold.*

Definition 14. An *oriented manifold* is a manifold with this class of orientations of its atlases, that is, with a fixed orientation on the manifold.

Thus orienting the manifold means exhibiting (by some means or other) a certain orientation class of atlases on it. To do this, for example, it suffices to exhibit any specific orienting atlas from the orientation class.

Various methods used in practice to define an orientation of manifolds embedded in $\mathbb{R}^n$ are described in Sects. 12.2 and 12.3.

Proposition 3. *A connected manfold is either nonorientable or admits exactly two orientations.*

Proof. Let A and $\widetilde{A}$ be two orienting atlases of the manifold M with diffeomorphic transitions from the local coordinates of charts of one to charts of the other. Assume that there is a point $p_0 \in M$ and two charts of these atlases whose ranges U_{i_0} and $\widetilde{U}_{i_0}$ contain p_0; and suppose the Jacobian of the change of coordinates of the charts at points of the parameter space corresponding to the point p_0 is positive. We shall show that then for every point $p \in M$ and any charts of the atlases A and $\widetilde{A}$ whose ranges contain p the Jacobian of the coordinate transformation at corresponding coordinate points is also positive.

We begin by making the obvious observation that if the Jacobian of the transformation is positive (resp. negative) at the point p for any pair of charts containing p in the atlases A and $\widetilde{A}$, then it is positive (resp. negative) at p for any such pair of charts, since inside each given atlas the coordinate transformations occur with positive Jacobian, and the Jacobian of a composition of two mappings is the product of the Jacobians of the individual mappings.

Now let E be the subset of M consisting of the points $p \in M$ at which the coordinate transformations from the charts of one atlas to those of the other have positive Jacobian.

The set E is nonempty, since $p_0 \in E$. The set E is open in M. Indeed, for every point $p \in E$ there exist ranges U_i and $\widetilde{U}_j$ of certain charts of the atlases A and $\widetilde{A}$ containing p. The sets U_i and $\widetilde{U}_j$ are open in M, so that the set $U_i \cap \widetilde{U}_j$ is open in M. On the connected component of the set $U_i \cap \widetilde{U}_j$ containing p, which is open in $U_i \cap \widetilde{U}_j$ and in M, the Jacobian of the transformation cannot change sign without vanishing at some point. That is, in some neighborhood of p the Jacobian remains positive, which proves that E is open. But E is also closed in M. This follows from the continuity of the Jacobian of a diffeomorphism and the fact that the Jacobian of a diffeomorphism never vanishes.

Thus E is a non-empty open-closed subset of the connected set M. Hence $E = M$, and the atlases A and $\widetilde{A}$ define the same orientation on M.

Replacing one coordinate, say t^1 by $-t^1$ in every chart of the atlas A, we obtain the orienting atlas $-A$ belonging to a different orientation class. Since the Jacobians of the coordinate transformations from an arbitrary chart to the charts of A and $-A$ have opposite signs, every atlas that orients M is equivalent either to A or to $-A$. □

Definition 15. A finite sequence of charts of a given atlas will be called a *chain of charts* if the ranges of any pair of charts having adjacent indices have a non-empty intersection ($U_i \cap U_{i+1} \neq \varnothing$).

Definition 16. A chain of charts is *contradictory* or *disorienting* if the Jacobian of the coordinate transformation from each chart in the chain to the next is positive and the ranges of the first and last charts of the chain intersect, but the coordinate transformation from the last to the first has negative Jacobian.

Proposition 4. *A manifold is orientable if and only if there does not exist a contradictory chain of charts on it.*

Proof. Since every manifold decomposes into connected components whose orientations can be defined independently, it suffices to prove Proposition 4 for a connected manifold M.

Necessity. Suppose the connected manifold M is orientable and A is an atlas defining an orientation. From what has been said and Proposition 3, every smooth local chart of the manifold M connected with the charts of the atlas A is either consistent with all the charts of A or consistent with all the charts of $-A$. This can easily be seen from Proposition 3 itself, if we resrict charts of A to the range of the chart we have taken, which can be regarded as a connected manifold oriented by one chart. It follows from this that there is no contradictory chain of charts on M.

Sufficiency. It follows from Definition 1 that there exists an atlas on the manifold consisting of a finite or countable number of charts. We take such an atlas A and number its charts. Consider the chart (U_1, φ_1) and any chart (U_i, φ_i) such that $U_1 \cap U_i \neq \varnothing$. Then the Jacobians of the coordinate transformations φ_{1i} and φ_{i1} are either everywhere negative or everywhere positive in their domains of definition. The Jacobians cannot have values of different signs, since otherwise one could exhibit connected subsets U_- and U_+ in $U_1 \cup U_i$ where the Jacobian is negative and positive respectively, and the chain of charts (U_1, φ_1), (U_+, φ_1), (U_i, φ_i), (U_-, φ_i) would be contradictory.

Thus, changing the sign of one coordinate if necessary in the chart (U_i, φ_i), we could obtain a chart with the same range U_i and consistent with (U_1, φ_1). After that procedure, two charts (U_i, φ_i) and (U_j, φ_j) such that $U_1 \cap U_i \neq \varnothing$, $U_1 \cap U_j \neq \varnothing$, $U_i \cap U_j \neq \varnothing$ are themselves consistent: otherwise we would have constructed a contradictory chain of three charts.

Thus, all the charts of an atlas whose ranges intersect U_1 can now be considered consistent with one another. Taking each of those charts now as the standard, one can adjust the charts of the atlas not covered in the first stage so that they are consistent. No contradictions arise when we do this, since by hypothesis, there are no contradictory chains on the manifold. Continuing this process and taking account of the connectedness of the manifold, we construct on it an atlas consisting of pairwise consistent charts, which proves the orientability of the manifold. □

This criterion for orientability of the manifold, like the considerations used in its proof, can be applied to the study of specific manifolds. Thus, the manifold $\mathbb{RP}^1$ studied in Example 12 is orientable. From the atlas shown there it is easy to obtain an orienting atlas of $\mathbb{RP}^1$. To do this, it suffices to reverse the sign of the local coordinates of one of the two charts constructed there. However, the orientability of the projective line $\mathbb{RP}^1$ obviously also follows from the fact that the manifold $\mathbb{RP}^1$ is homoeomorphic to a circle.

The projective plane $\mathbb{RP}^2$ is nonorientable: every pair of charts in the atlas constructed in Example 13 is such that the coordinate transformations have domains of positivity and domains of negativity of the Jacobian. As we saw in the proof of Proposition 4, it follows from this that a contradictory chain of charts on $\mathbb{RP}^2$ exists.

For the same reason the manifold considered in Example 14 is nonorientable, which, as was noted, is homeomorphic to a Möbius band.

Proposition 5. *The boundary of an orientable smooth n-dimensional manifold is an orientable $(n-1)$-dimensional manifold admitting a structure of the same smoothness as the original manifold.*

Proof. The proof of Proposition 5 is a verbatim repetition of the proof of the analogous Proposition 2 of Subsect. 12.3.2 for surfaces embedded in $\mathbb{R}^n$. □

Definition 17. If $A(M) = \{(H^n, \varphi_i, U_i)\} \cup \{(\mathbb{R}^n, \varphi_j, U_j)\}$ is an atlas that orients the manifold M, then the charts $A(\partial M) = \{(\mathbb{R}^{n-1}, \varphi_i\big|_{\partial H^n = \mathbb{R}^{n-1}}, \partial U_i)\}$ provide an orienting atlas for the boundary ∂M of M. The orientation of the boundary defined by this atlas is called the *orientation of the boundary induced by the orientation of the manifold.*

Important techniques for defining the orientation of a surface embedded in $\mathbb{R}^n$ and the induced orientation of its boundary, which are frequently used in practice, were described in detail in Sects. 12.2 and 12.3.

15.2.4 Partitions of Unity and the Realization of Manifolds as Surfaces in $\mathbb{R}^n$

In this subsection we shall describe a special construction called a *partition of unity.* This construction is often the basic device for reducing global problems to local ones. Later on we shall demonstrate it in deriving Stokes' formula on a manifold, but here we shall use the partition of unity to clarify the possibility of realizing any manifold as a surface in $\mathbb{R}^n$ of sufficiently high dimension.

Lemma. *One can construct a function $f \in C^{(\infty)}(\mathbb{R}, \mathbb{R})$ on $\mathbb{R}$ such that $f(x) \equiv 0$ for $|x| \geq 3$, $f(x) \equiv 1$ for $|x| \leq 1$, and $0 < f(x) < 1$ for $1 < |x| < 3$.*

Proof. We shall construct one such function using the familiar function $g(x) = \begin{cases} e^{(-1/x^2)} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$ Previously (see Exercise 2 of Sect. 5.2) we verified that $g \in C^{(\infty)}(\mathbb{R}, \mathbb{R})$ by showing that $g^{(n)}(0) = 0$ for every value $n \in \mathbb{N}$.

In such a case the nonnegative function

$$G(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

also belongs to $C^{(\infty)}(\mathbb{R}, \mathbb{R})$, and along with it the function

$$F(x) = \int_{-\infty}^{x} G(t)\,dt \Big/ \int_{-\infty}^{+\infty} G(t)\,dt$$

belongs to this class, since $F'(x) = G(x) \Big/ \int\limits_{-\infty}^{\infty} G(t)\,dt$.

The function F is strictly increasing on $[-1,1]$, $F(x) \equiv 0$ for $x \le -1$, and $F(x) \equiv 1$ for $x \ge 1$.

We can now take the required function to be

$$f(x) = F(x+2) + F(-x-2) - 1 . \ \square$$

Remark. If $f : \mathbb{R} \to \mathbb{R}$ is the function constructed in the proof of the lemma, then the function

$$\theta(x^1, \dots, x^n) = f(x^1 - a^1) \cdot \ldots \cdot f(x^n - a^n)$$

defined in $\mathbb{R}^n$ is such that $\theta \in C^{(\infty)}(\mathbb{R}^n, \mathbb{R})$, $0 \le \theta \le 1$, at every point $x \in \mathbb{R}^n$, $\theta(x) \equiv 1$ on the interval $I(a) = \{x \in \mathbb{R}^n \,\big|\, |x^i - a^i| \le 1,\ i = 1, \dots, n\}$, and the support $\operatorname{supp}\theta$ of the function θ is contained in the interval $\tilde{I}(a) = \{x \in \mathbb{R}^n \mid |x^i - a^i| \le 3,\ i = 1, \dots, n\}$.

Definition 18. Let M be a $C^{(k)}$-manifold and X a subset of M. The system $E = \{e_\alpha,\ \alpha \in A\}$ of functions $e_\alpha \in C^{(k)}(M, \mathbb{R})$ is a $C^{(k)}$ *partition of unity* on X if

1^0 $0 \le e_\alpha(x) \le 1$ for every function $e_\alpha \in E$ and every $x \in M$;

2^0 each point $x \in X$ has a neighborhood $U(x)$ in M such that all but a finite number of functions of E are identically zero on $U(x)$;

3^0 $\sum\limits_{e_\alpha \in E} e_\alpha(x) \equiv 1$ on X.

We remark that by condition 2^0 only a finite number of terms in this last sum are nonzero at each point $x \in X$.

Definition 19. Let $\mathcal{O} = \{o_\beta,\ \beta \in B\}$ be an open covering of $X \subset M$. We say that *the partition of unity* $E = \{e_\alpha,\ \alpha \in A\}$ on X is *subordinate to the covering* $\mathcal{O}$ if the support of each function in the system E is contained in at least one of the sets of the system $\mathcal{O}$.

Proposition 6. *Let $\{(U_i, \varphi_i),\ i = 1, \dots, m\}$ be a finite set of charts of some $C^{(k)}$ atlas of the manifold M, whose ranges U_i, $i = 1, \dots, m$, form a covering of a compact set $K \subset M$. Then there exists a $C^{(k)}$ partition of unity on K subordinate to the covering $\{U_i,\ i = 1, \dots, m\}$.*

Proof. For any point $x_0 \in K$ we first carry out the following construction. We choose successively a domain U_i containing x_0 corresponding to a chart $\varphi_i : \mathbb{R}^n \to U_i$ (or $\varphi_i : H^n \to U_i$), the point $t_0 = \varphi_i^{-1}(x_0) \in \mathbb{R}^n$ (or H^n), the function $\theta(t - t_0)$ (where $\theta(t)$ is the function shown in the remark to the lemma), and the restriction θ_{t_0} of $\theta(t - t_0)$ to the parameter domain of φ_i.

Let I_{t_0} be the intersection of the unit cube centered at $t_0 \in \mathbb{R}^n$ with the parameter domain of φ_i. Actually θ_{t_0} differs from $\theta(t - t_0)$ and I_{t_0} differs from the corresponding unit cube only when the parameter domain of the chart φ_i is the half-space H^n. The open sets $\varphi_i(I_t)$ constructed at each point $x \in K$ and the point $t = \varphi_i^{-1}(x)$, taken for all admissible values of $i = 1, 2, \dots, m$, form an open covering of the compact set K. Let $\{\varphi_{i_j}(I_{t_j}), j = 1, 2, \dots, l\}$ be a finite covering of K extracted from it. It is obvious that $\varphi_{i_j}(I_{t_j}) \subset U_{i_j}$. We define on U_{i_j} the function $\widetilde{\theta}_i(x) = \theta_{t_j} \circ \varphi_{t_j}^{-1}(x)$. We then extend $\widetilde{\theta}_j(x)$ to the entire manifold M by setting the function equal to zero outside U_{i_j}. We retain the previous notation $\widetilde{\theta}_j$ for this function extended to M. By construction $\widetilde{\theta}_j \in C^{(k)}(M, \mathbb{R})$, $\operatorname{supp} \widetilde{\theta}_j \subset U_{i_j}$, $0 \le \widetilde{\theta}_j(x) \le 1$ on M, and $\widetilde{\theta}_j(x) \equiv 1$ on $\varphi_{i_j}(I_{t_j}) \subset U_{i_j}$. Then the functions $e_1(x) = \widetilde{\theta}_1(x)$, $e_2(x) = \widetilde{\theta}_2(x)(1 - \widetilde{\theta}_1(x)), \dots, e_l(x) = \widetilde{\theta}_l(x) \cdot (1 - \widetilde{\theta}_{l-1}(x)) \cdot \ldots \cdot (1 - \widetilde{\theta}_1(x))$ form the required partition of unity. We shall verify only that $\sum_{j=1}^{l} e_j(x) \equiv 1$ on K, since the system of functions $\{e_1, \dots, e_l\}$ obviously satisfies the other conditions required of a partition of unity on K subordinate to the covering $\{U_{i_1}, \dots, U_{i_l}\} \subset \{U_i, i = 1, \dots, m\}$. But

$$1 - \sum_{j=1}^{l} e_j(x) = (1 - \widetilde{\theta}_1(x)) \cdot \ldots \cdot (1 - \widetilde{\theta}_1(x)) \equiv 0 \text{ on } K ,$$

since each point $x \in K$ is covered by some set $\varphi_{i_j}(I_{t_j})$ on which the corresponding function $\widetilde{\theta}_j$ is identically equal to 1. □

Corollary 1. *If M is a compact manifold and A a $C^{(k)}$ atlas on M, then there exists a finite partition of unity $\{e_1, \dots, e_l\}$ on M subordinate to a covering of the manifold by the ranges of the charts of A.*

Proof. Since M is compact, the atlas A can be regarded as finite. We now have the hypotheses of Proposition 6, if we set $K = M$ in it. □

Corollary 2. *For every compact set K contained in a manifold M and every open set $G \subset M$ containing K, there exists a function $f : M \to \mathbb{R}$ with smoothness equal to that of the manifold and such that $f(x) \equiv 1$ on K and $\operatorname{supp} f \subset G$.*

Proof. Cover each point $x \in K$ by a neighborhood $U(x)$ contained in G and inside the range of some chart of the manifold M. From the open covering $\{U(x), x \in K\}$ of the compact set K extract a finite covering, and construct a partition of unity $\{e_1, \dots, e_l\}$ on K subordinate to it. The function $f = \sum_{i=1}^{l} e_i$ is the one required. □

Corollary 3. *Every (abstractly defined) compact smooth n-dimensional manifold M is diffeomorphic to some compact smooth surface contained in $\mathbb{R}^N$ of sufficiently large dimension N.*

Proof. So as not to complicate the idea of the proof with inessential details, we carry it out for the case of a compact manifold M without boundary. In that case there is a finite smooth atlas $A = \{\varphi_i : I \to U_i,\ i = 1, \dots, m\}$ on M, where I is an open n-dimensional cube in $\mathbb{R}^n$. We take a slightly smaller cube I' such that $I' \subset I$ and the set $\{U_i' = \varphi_i(I'),\ i = 1, \dots, m\}$ still forms a covering of M. Setting $K = I'$, $G = I$, and $M = \mathbb{R}^n$ in Corollary 2, we construct a function $f \in C^{(\infty)}(\mathbb{R}^n, \mathbb{R})$ such that $f(t) \equiv 1$ for $t \in I'$ and $\operatorname{supp} f \subset I$.

We now consider the coordinate functions $t_i^1(x), \dots, t_i^n(x)$ of the mappings $\varphi_i^{-1} : U_I \to I$, $i = 1, \dots, m$, and use them to introduce the following function on M:

$$y_i^k(x) = \begin{cases} (f \circ \varphi_i^{-1})(x) \cdot t_i^k(x) & \text{for } x \in U_i\,, \\ 0 & \text{for } x \notin U_i\,, \end{cases}$$
$$i = 1, \dots, m\,; \quad k = 1, \dots, n\,.$$

At every point $x \in M$ the rank of the mapping $M \ni x \mapsto y(x) = (y_1^1, \dots, y_1^n, \dots y_m^1, \dots, y_m^n)(x) \in \mathbb{R}^{m \cdot n}$ is maximal and equal to n. Indeed, if $x \in U_i'$, then $\varphi_i^{-1}(x) = t \in I'$, $f \circ \varphi_i^{-1}(x) = 1$, and $y_i^k(\varphi_i(t)) = t_i^k$, $k = 1, \dots, n$.

If finally, we consider the mapping $M \ni x \mapsto Y(x) = (y(x), f \circ \varphi_1^{-1}(x), \dots, f \circ \varphi_m^{-1}(x)) \in \mathbb{R}^{m \cdot n + m}$, setting $f \circ \varphi_i^{-1}(x) \equiv 0$ outside U_i, $i = 1, \dots, m$, then this mapping, on the one hand will obviously have the same rank n as the mapping $x \mapsto y(x)$; on the other hand it will be demonstrably a one-to-one mapping of M onto the image of M in $\mathbb{R}^{m \cdot n + m}$. Let us verify this last assertion. Let p, q be different points of M. We find a domain U_i' from the system $\{U_i',\ i = 1, \dots, m\}$ covering M that contains the point p. Then $f \circ \varphi_i^{-1}(p) = 1$. If $f \circ \varphi_i^{-1}(q) < 1$, then $Y(p) \neq Y(q)$. If $f \circ \varphi_i^{-1}(q) = 1$, then $p, q \in U_i$, $y_i^k(p) = t^k(p)$, $y_i^k(q) = t^k(q)$, and $t_i^k(p) \neq t_i^k(q)$ for at least one value of $k \in \{1, \dots, n\}$. That is, $Y(p) \neq Y(q)$ in this case. □

For information on the general Whitney embedding theorem for an arbitrary manifold as a surface in $\mathbb{R}^n$ the reader may consult the specialized geometric literature.

15.2.5 Problems and Exercises

1. Verify that the object (a *manifold*) introduced by Definition 1 does not change if we require only that each point $x \in M$ have a neighborhood $U(x) \subset M$ homeomorphic to an open subset of the half-space H^n.

2. Show that

a) the manifold $GL(n, \mathbb{R})$ of Example 6 is noncompact and has exactly two connected components;

b) the manifold $SO(n, \mathbb{R})$ (see Example 7) is connected;

c) the manifold $O(n, \mathbb{R})$ is compact and has exactly two connected components.

3. Let (M, A) and $(\widetilde{M}, \widetilde{A})$ be manifolds with smooth structures of the same degree of smoothness $C^{(k)}$ on them. The smooth manifolds (M, A) and $(\widetilde{M}, \widetilde{A})$ (*smooth structures*) are considered *isomorphic* if there exists a $C^{(k)}$ mapping $f : M \to \widetilde{M}$ having a $C^{(k)}$ inverse $f^{-1} : \widetilde{M} \to M$ in the atlases $A, \widetilde{A}$.

a) Show that all structures of the same smoothness on $\mathbb{R}^1$ are isomorphic.

b) Verify the assertions made in Example 11, and determine whether they contradict a).

c) Show that on the circle S^1 (the one-dimensional sphere) any two $C^{(\infty)}$ structures are isomorphic. We note that this assertion remains valid for spheres of dimension not larger than 6, but on S^7, as Milnor[4] has shown, there exist nonisomorphic $C^{(\infty)}$ structures.

4. Let S be a subset of an n-dimensional manifold M such that for every point $x_0 \in S$ there exists a chart $x = \varphi(t)$ of the manifold M whose rauge U contains x_0, and the k-dimensional surface defined by the relations $t^{k+1} = 0, \dots, t^n = 0$ corresponds to the set $S \cap U$ in the parameter domain $t = (t^1, \dots, t^n)$ of φ. In this case S is called a *k-dimensional submanifold of M*.

a) Show that a k-dimensional manifold structure naturally arises on S, induced by the structure of M and having the same smoothness as the manifold M.

b) Verify that the k-dimensional surfaces S in $\mathbb{R}^n$ are precisely the k-dimensional submanifolds of $\mathbb{R}^n$.

c) Show that under a smooth homeomorphic mapping $f : \mathbb{R}^1 \to T^2$ of the line $\mathbb{R}^1$ into the torus T^2 the image $f(\mathbb{R}^1)$ may be an everywhere dense subset of T^2 aud in that case will not be a one-dimensional submanifold of the torus, although it will be an abstract one-dimensional manifold.

d) Verify that the extent of the concept "submanifold" does not change if we consider $S \subset M$ a k-dimensional submanifold of the n-dimensional manifold M when there exists a local chart of the manifold M whose range contains x_0 for every point $x_0 \in S$ and some k-dimensional surface of the space $\mathbb{R}^n$ corresponds to the set $S \cap U$ in the parameter domain of the chart.

5. Let X be a Hausdorff topological space (manifold) and G the group of homeomorphic transformations of X. The group G is a *discrete group of transformations of X* if for every two (possibly equal) points $x_1, x_2 \in X$ there exist neighborhoods U_1 and U_2 of them respectively, such that the set $\{g \in G \,\big|\, g(U_1) \cap U_2 \neq \varnothing\}$ is finite.

a) It follows from this that the *orbit* $\{g(x) \in X \,|\, g \in G\}$ of every point $x \in X$ is discrete, and the *stabilizer* $G_x = \{g \in G \,|\, g(x) = x\}$ of every point $x \in X$ is finite.

b) Verify that if G is a group of isometries of a metric space, having the two properties in a), then G is a discrete group of transformations of X.

c) Introduce the natural topological space (manifold) structure on the set X/G of orbits of the discrete group G.

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[4] J. Milnor (b. 1931) – one of the most outstanding modern American mathematicians; his main works are in algebraic topology and the topology of manifolds.

d) A closed subset F of the topological space (manifold) X with a discrete group G of transformations is a *fundamental domain of the group* G if it is the closure of an open subset of X and the sets $g(F)$, where $g \in G$, have no interior points in common and form a locally finite covering of X. Show using Examples 8–10 how the quotient space X/G (of orbits) of the group G can be obtained from F by "gluing" certain boundary points.

6. a) Using the construction of Examples 12 and 13, construct n-dimensional projective space $\mathbb{RP}^n$.

b) Show that $\mathbb{RP}^n$ is orientable if n is odd and nonorientable if n is even.

c) Verify that the manifolds $SO(3, \mathbb{R})$ and $\mathbb{RP}^3$ are homeomorphic.

7. Verify that the manifold constructed in Example 14 is indeed homeomorphic to the Möbius band.

8. a) A *Lie group*[5] is a group G endowed with the structure of an analytic manifold such that the mappings $(g_1, g_2) \mapsto g_1 \cdot g_2$ and $g \mapsto g^{-1}$ are analytic mappings of $G \times G$ and G into G. Show that the manifolds in Examples 6 and 7 are Lie groups.

b) A *topological group* (or *continuous group*) is a group G endowed with the structure of a topological space such that the group operations of multiplication and inversion are continuous as mappings $G \times G \to G$, and $G \to G$ in the topology of G. Using the example of the group $\mathbb{Q}$ of rational numbers show that not every topological group is a Lie group.

c) Show that every Lie group is a topological group in the sense of the definition given in b).

d) It has been proved[6] that every topological group G that is a manifold is a Lie group (that is, as a manifold G admits an analytic structure in which the group becomes a Lie group). Show that every group manifold (that is, every Lie group) is an orientable manifold.

9. A system of subsets of a topological space is *locally finite* if each point of the space has a neighborhood intersecting only a finite number of sets in the system. In particular, one may speak of a locally finite covering of a space.

A system of sets is said to be a *refinement* of a second system if every set of the first system is contained in at least one of the sets of the second system. In particular it makes sense to speak of one covering of a set being a refinement of another.

a) Show that every open covering of $\mathbb{R}^n$ has a locally finite refinement.

b) Solve problem a) with $\mathbb{R}^n$ replaced by an arbitrary manifold M.

c) Show that there exists a partition of unity on $\mathbb{R}^n$ subordinate to any preassigned open covering of $\mathbb{R}^n$.

d) Verify that assertion c) remains valid for an arbitrary manifold.

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[5] S. Lie (1842–1899) – outstanding Norwegian mathematician, creator of the theory of continuous groups (Lie groups), which is now of fundamental importance in geometry, topology, and the mathematical methods of physics; one of the winners of the International Lobachevskii Prize (awarded in 1897 for his work in applying group theory to the foundations of geometry).

[6] This is the solution to Hilbert's fifth problem.

15.3 Differential Forms and Integration on Manifolds

15.3.1 The Tangent Space to a Manifold at a Point

We recall that to each smooth path $\mathbb{R} \ni t \overset{\gamma}{\longmapsto} x(t) \in \mathbb{R}^n$ (a motion in $\mathbb{R}^n$) passing through the point $x_0 = x(t_0) \in \mathbb{R}^n$ at time t_0 we have assigned the instantaneous velocity vector $\xi = (\xi^1, \dots, \xi^n)$: $\xi(t) = \dot{x}(t) = (\dot{x}^1, \dots, \dot{x}^n)(t_0)$. The set of all such vectors ξ attached to the point $x_0 \in \mathbb{R}^n$ is naturally identified with the arithmetic space $\mathbb{R}^n$ and is denoted $T\mathbb{R}^n_{x_0}$ (or $T_{x_0}(\mathbb{R}^n)$). In $T\mathbb{R}^n_{x_0}$ one introduces the same vector operations on elements $\xi \in T\mathbb{R}^n_{x_0}$ as on the corresponding elements of the vector space $\mathbb{R}^n$. In this way a vector space $T\mathbb{R}^n_{x_0}$ arises, called the *tangent space to* $\mathbb{R}^n$ *at the point* $x_0 \in \mathbb{R}^n$.

Forgetting about motivation and introductory considerations, we can now say that formally $T\mathbb{R}^n_{x_0}$ is a pair $(x_0, \mathbb{R}^n)$ consisting of a point $x_0 \in \mathbb{R}^n$ and a copy of the vector space $\mathbb{R}^n$ attached to it.

Now let M be a smooth n-dimensional manifold with an atlas A of at least $C^{(1)}$ smoothness. We wish to define a tangent vector ξ and a tangent space TM_{p_0} to the manifold M at a point $p_0 \in M$.

To do this we use the interpretation of the tangent vector as the instantaneous velocity of a motion. We take a smooth path $\mathbb{R}^n \ni t \overset{\gamma}{\longmapsto} p(t) \in M$ on the manifold M passing through the point $p_0 = p(t_0) \in M$ at time t_0. The parameters of charts (that is, local coordinates) of the manifold M will be denoted by the letter x here, with the subscript of the corresponding chart and a superscript giving the number of the coordinate. Thus, in the parameter domain of each chart (U_i, φ_i) whose range U_i contains p_0, the path $t \overset{\gamma_i}{\longmapsto} \varphi_i^{-1} \circ p(t) = x_i(t) \in \mathbb{R}^n$ (or H^n) corresponds to the path γ. This path is smooth by definition of the smooth mapping $\mathbb{R} \ni t \overset{\gamma}{\longmapsto} p(t) \in M$.

Thus, in the parameter domain of the chart (U_i, φ_i), where φ_i is a mapping $p = \varphi_i(x_i)$, there arises a point $x_i(t_0) = \varphi_i^{-1}(p_0)$ and a vector $\xi_i = \dot{x}_i(t_0) \in T\mathbb{R}^n_{x_i(t_0)}$. In another such chart (U_j, φ_j) these objects will be respectively the point $x_j(t_0) = \varphi_j^{-1}(p_0)$ and the vector $\xi_j = \dot{x}_j(t_0) \in T\mathbb{R}^n_{x_j(t_0)}$. It is natural to regard these as the coordinate expressions in different charts of what we would like to call a tangent vector ξ to the manifold M at the point $p_0 \in M$.

Between the coordinates x_i and x_j there are smooth mutually inverse transition mappings

$$x_i = \varphi_{ji}(x_j) , \quad x_j = \varphi_{ij}(x_i) , \tag{15.30}$$

as a result of which the pairs $(x_i(t_0), \xi_i)$, $(x_j(t_0), \xi_j)$ turn out to be connected by the relations

$$x_i(t_0) = \varphi_{ji}\big(x_j(t_0)\big) , \qquad x_j(t_0) = \varphi_{ij}\big(x_i(t_0)\big) , \tag{15.31}$$

$$\xi_i = \varphi'_{ji}\big(x_j(t_0)\big)\xi_j , \qquad \xi_j = \varphi'_{ij}\big(x_i(t_0)\big)\xi_i . \tag{15.32}$$

Equality (15.32) obviously follows from the formulas

$$\dot{x}_i(t) = \varphi'_{ji}(x_j(t))\dot{x}_j(t) , \qquad \dot{x}_j(t) = \varphi'_{ij}(x_i(t))\dot{x}_i(t) ,$$

obtained from (15.30) by differentiation.

Definition 1. We shall say that a *tangent vector* ξ *to the manifold* M *at the point* $p \in M$ is defined if a vector ξ_i is fixed in each space $T\mathbb{R}^n_{x_i}$ tangent to $\mathbb{R}^n$at the point x_i corresponding to p in the parameter domain of a chart (U_i, φ_i), where $U_i \ni p$, in such a way that (15.32) holds.

If the elements of the Jacobian matrix φ'_{ji} of the mapping φ_{ji} are written out explicitly as $\frac{\partial x_i^k}{\partial x_j^m}$, we find the following explicit form for the connection between the two coordinate representations of a given vector ξ:

$$\xi_i^k = \sum_{m=1}^{n} \frac{\partial x_i^k}{\partial x_j^m}\xi_j^m , \qquad k = 1, 2, \dots, n , \tag{15.33}$$

where the partial derivatives are computed at the point $x_j = \varphi_j^{-1}(p)$ corresponding to p.

We denote by TM_p the set of all tangent vectors to the manifold M at the point $p \in M$.

Definition 2. If we introduce a vector-space structure on the set TM_p by identifying TM_p with the corresponding space $T\mathbb{R}^n_{x_i}$ (or $TH^n_{x_i}$), that is, the sum of vectors in TM_p is regarded as the vector whose coordinate representation in $T\mathbb{R}^n_{x_i}$ (or $TH^n_{x_i}$) corresponds to the sum of the coordinate representations of the terms, and multiplication of a vector by a scalar is defined analogously, the vector space so obtained is usually denoted either TM_p or T_pM, and is called the *tangent space to the manifold* M *at the point* $p \in M$.

It can be seen from formulas (15.32) and (15.33) that the vector-space structure introduced in TM_p is independent of the choice of individual chart, that is, Definition 2 is unambiguous in that sense.

Thus we have now defined the tangent space to a manifold. There are various interpretations of a tangent vector and the tangent space (see Problem 1). For example, one such interpretation is to identify a tangent vector with a linear functional. This identification is based on the following observation, which we make in $\mathbb{R}^n$.

Each vector $\xi \in T\mathbb{R}^n_{x_0}$ is the velocity vector corresponding to some smooth path $x = x(t)$, that is, $\xi = \dot{x}(t)\big|_{t=t_0}$ with $x_0 = x(t_0)$. This makes it possible to define the derivative $D_\xi f(x_0)$ of a smooth function f defined on $\mathbb{R}^n$ (or in a neighborhood of x_0) with respect to the vector $\xi \in T\mathbb{R}^n_{x_0}$. To be specific,

$$D_\xi f(x_0) := \frac{\mathrm{d}}{\mathrm{d}t}(f \circ x)(t)\big|_{t=t_0} , \tag{15.34}$$

that is,

$$D_\xi f(x_0) = f'(x_0)\xi \,, \tag{15.35}$$

where $f'(x_0)$ is the tangent mapping to f (the differential of f) at a point x_0.

The functional $D_\xi : C^{(1)}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ assigned to the vector $\xi \in T\mathbb{R}^n_{x_0}$ by the formulas (15.34) and (15.35) is obviously linear with respect to f. It is also clear from (15.35) that for a fixed function f the quantity $D_\xi f(x_0)$ is a linear function of ξ, that is, the sum of the corresponding linear functionals corresponds to a sum of vectors, and multiplication of a functional D_ξ by a number corresponds to multiplying the vector ξ by the same number. Thus there is an isomorphism between the vector space $T\mathbb{R}^n_{x_0}$ and the vector space of corresponding linear functionals D_ξ. It remains only to define the linear functional D_ξ by exhibiting a set of characteristic properties of it, in order to obtain a new interpretation of the tangent space $T\mathbb{R}^n_{x_0}$, which is of course isomorphic to the previous one.

We remark that, in addition to the linearity indicated above, the functional D_ξ possesses the following property:

$$D_\xi(f \cdot g)(x_0) = D_\xi f(x_0) \cdot g(x_0) + f(x_0) \cdot D_\xi g(x_0) \,. \tag{15.36}$$

This is the law for differentiating a product.

In differential algebra an additive mapping $a \mapsto a'$ of a ring A satisfying the relation $(a \cdot b)' = a' \cdot b + a \cdot b'$ is called *derivation* (more precisely *derivation of the ring* A) Thus the functional $D_\xi : C^{(1)}(\mathbb{R}^n, \mathbb{R})$ is a derivation of the ring $C^{(1)}(\mathbb{R}^n, \mathbb{R})$. But D_ξ is also linear relative to the vector-space structure of $C^{(1)}(\mathbb{R}^n, \mathbb{R})$.

One can verify that a linear functional $l : C^{(\infty)}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ possessing the properties

$$l(\alpha f + \beta g) = \alpha l(f) + \beta l(g) \,, \quad \alpha, \beta \in \mathbb{R} \,, \tag{15.37}$$

$$l(f \cdot g) = l(f) g(x_0) + f(x_0) l(g) \,, \tag{15.38}$$

has the form D_ξ, where $\xi \in T\mathbb{R}^n_{x_0}$. Thus the tangent space $T\mathbb{R}^n_{x_0}$ to $\mathbb{R}^n$ at x_0 can be interpreted as a vector space of functionals (derivations) on $C^{(\infty)}(\mathbb{R}^n, \mathbb{R})$ satisfying conditions (15.37) and (15.38).

The functions $D_{e_k} f(x_0) = \frac{\partial}{\partial x^k} f(x)\big|_{x=x_0}$ that compute the corresponding partial derivative of the function f at x_0 correspond to the basis vectors $e_1, \dots, e_n$ of the space $T\mathbb{R}^n_{x_0}$. Thus, under the functional interpretation of $T\mathbb{R}^n_{x_0}$ one can say that the functionals $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}\big|_{x=x_0}$ form a basis of $T\mathbb{R}^n_{x_0}$.

If $\xi = (\xi^1, \dots, \xi^n) \in T\mathbb{R}^n_{x_0}$, then the operator D_ξ corresponding to the vector ξ has the form $D_\xi = \xi^k \frac{\partial}{\partial x^k}$.

In a completely analogous manner the tangent vector ξ to an n-dimensional $C^{(\infty)}$ manifold M at a point $p_0 \in M$ can be interpreted (or defined) as the element of the space of derivations l on $C^{(\infty)}(M, \mathbb{R})$ having properties (15.37) and (15.38), x_0 of course being replaced by p_0 in relation (15.38), so that the functional l is connected with precisely the point $p_0 \in M$. Such a definition of the tangent vector ξ and the tangent space TM_{p_0} does not formally require the invocation of any local coordinates, and in that sense it is obviously invariant. In coordinates $(x^1, \dots, x^n)$ of a local chart (U_i, φ_i) the operator l has the form $\xi_i^1 \frac{\partial}{\partial x_i^1} + \cdots + \xi_i^n \frac{\partial}{\partial x_i^n} = D_{\xi_i}$. The numbers $(\xi_i^1, \dots, \xi_i^n)$ are naturally called the *coordinates of the tangent vector* $l \in TM_{p_0}$ in coordinates of the chart (U_i, φ_i). By the laws of differentiation, the coordinate representations of the same functional $l \in TM_{p_0}$ in the charts (U_i, φ_i), (U_j, φ_j) are connected by the relations

$$\sum_{k=1}^{n} \xi_i^k \frac{\partial}{\partial x_i^k} = \sum_{m=1}^{n} \xi_j^m \frac{\partial}{\partial x_j^m} = \sum_{k=1}^{n} \Big(\sum_{m=1}^{n} \frac{\partial x_i^k}{\partial x_j^m} \xi_j^m \Big) \frac{\partial}{\partial x_i^k}\,, \qquad (15.33')$$

which of course duplicate (15.33).

15.3.2 Differential Forms on a Manifold

Let us now consider the space T^*M_p conjugate to the tangent space TM_p, that is, T^*M_p is the space of real-valued linear functionals on TM_p.

Definition 3. The space T^*M_p conjugate to the tangent space TM_p to the manifold M at the point $p \in M$ is called the *cotangent space to M at p*.

If the manifold M is a $C^{(\infty)}$ manifold, $f \in C^{(\infty)}(M, \mathbb{R})$, and l_ξ is the derivation corresponding to the vector $\xi \in TM_0$, then for a fixed $f \in C^{(\infty)}(M, \mathbb{R})$ the mapping $\xi \mapsto l_\xi f$ will obviously be an element of the space T^*M_p. In the case $M = \mathbb{R}^n$ we obtain $\xi \mapsto D_\xi f(p) = f'(p)\xi$, so that the resulting mapping $\xi \mapsto l_\xi f$ is naturally called the differential of the function f at p, and is denoted by the usual symbol $\mathrm{d}f(p)$.

If $T\mathbb{R}^n_{\varphi_\alpha^{-1}(p)}$ (or $TH^n_{\varphi_\alpha^{-1}(p)}$ when $p \in \partial M$) is the space corresponding to the tangent space TM_p in the chart $(U_\alpha, \varphi_\alpha)$ on the manifold M, it is natural to regard the space $T^*\mathbb{R}^n_{\varphi_\alpha^{-1}(p)}$ conjugate to $T\mathbb{R}^j_{\varphi_\alpha^{-1}(p)}$ as the representative of the space T^*M_p in this local chart. In coordinates $(x_\alpha^1, \dots, x_\alpha^n)$ of a local chart $(U_\alpha, \varphi_\alpha)$ the dual basis $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ in the conjugate space corresponds to the basis $\left\{\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^n}\right\}$ of $T\mathbb{R}^n_{\varphi_\alpha^{-1}(p)}$ (or $TH^n_{\varphi_\alpha^{-1}(p)}$ if $p \in \partial M$). We recall that $\mathrm{d}x^i(\xi) = \xi^i$, so that $\mathrm{d}x^i\big(\frac{\partial}{\partial x^j}\big) = \delta^i_j$. The expressions for these dual bases in another chart (U_β, φ_β) may turn out to be not so simple, for $\frac{\partial}{\partial x_\beta^j} = \frac{\partial x_\alpha^i}{\partial x_\beta^j} \frac{\partial}{\partial x_\alpha^i}$, $\mathrm{d}x_\alpha^i = \frac{\partial x_\alpha^i}{\partial x_\beta^j} \mathrm{d}x_\beta^j$.

Definition 4. We say that a *differential form ω^m of degree m* is defined on an n-dimensional manifold M if a skew-symmetric form $\omega^m(p) : (TM_p)^m \to \mathbb{R}$ is defined on each tangent space TM_p to M, $p \in M$.

In practice this means only that a corresponding m-form $\omega_\alpha(x_\alpha)$, where $x_\alpha = \varphi_\alpha^{-1}(p)$, is defined on each space $T\mathbb{R}^n_{\varphi_\alpha^{-1}(p)}$ (or $TH^n_{\varphi_\alpha^{-1}(p)}$) corresponding to TM_0 in the chart $(U_\alpha, \varphi_\alpha)$ of the manifold M. The fact that two such forms $\omega_\alpha(x_\alpha)$ and $\omega_\beta(x_\beta)$ are representatives of the same form $\omega(p)$ can be expressed by the relation

$$\omega_\alpha(x_\alpha)\big((\xi_1)_\alpha, \dots, (\xi_m)_\alpha\big) = \omega_\beta(x_\beta)\big((\xi_1)_\beta, \dots, (\xi_m)\beta\big)\ , \tag{15.39}$$

in which x_α and x_β are the representatives of the point $p \in M$, and $(\xi_1)_\alpha, \dots, (\xi_m)_\alpha$ and $(\xi_1)_\beta, \dots, (\xi_m)_\beta$ are the coordinate representations of the vectors $\xi_1, \dots, \xi_m \in TM_p$ in the charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ respectively.

In more formal notation this means that

$$x_\alpha = \varphi_{\beta\alpha}(x_\beta)\ , \qquad x_\beta = \varphi_{\alpha\beta}(x_\alpha)\ , \tag{15.31$'$}$$

$$\xi_\alpha = \varphi'_{\beta\alpha}(x_\beta)\xi_\beta\ , \qquad \xi_\beta = \varphi'_{\alpha\beta}(x_\alpha)\xi_\alpha\ , \tag{15.32$'$}$$

where, as usual, $\varphi_{\beta\alpha}$ and $\varphi_{\alpha\beta}$ are respectively the functions $\varphi_\alpha^{-1} \circ \varphi_\beta$ and $\varphi_\beta^{-1} \circ \varphi_\alpha$ for the coordinate transitions, and the tangent mappings to them $\varphi'_{\beta\alpha} =: (\varphi_{\beta\alpha})_*$, $\varphi'_{\alpha\beta} =: (\varphi_{\alpha\beta})_*$ provide an isomorphism of the tangent spaces to $\mathbb{R}^n$ (or H^n) at the corresponding points x_α and x_β. As stated in Subsect. 15.1.3, the adjoint mappings $(\varphi'_{\beta\alpha})^* =: \varphi^*_{\beta\alpha}$ and $(\varphi'_{\alpha\beta})^* =: \varphi^*_{\alpha\beta}$ provide the transfer of the forms, and the relation (15.39) means precisely that

$$\omega_\alpha(x_\alpha) = \varphi^*_{\alpha\beta}(x_\alpha)\omega_\beta(x_\beta)\ , \tag{15.39$'$}$$

where α and β are indices (which can be interchanged).

The matrix (c_i^j) of the mapping $\varphi'_{\alpha\beta}(x_\alpha)$ is known: $(c_i^j) = \left(\frac{\partial x_\beta^j}{\partial x_\alpha^i}\right)(x_\alpha)$. Thus, if

$$\omega_\alpha(x_\alpha) = \sum_{1 \le i_1 < \dots < i_m \le n} a_{i_1, \dots, i_m}\, \mathrm{d}x_\alpha^{i_1} \wedge \dots \wedge \mathrm{d}x_\alpha^{i_m} \tag{15.40}$$

and

$$\omega_\beta(x_\beta) = \sum_{1 \le j_1 < \dots < j_m \le n} b_{j_1, \dots, j_m}\, \mathrm{d}x_\beta^{j_1} \wedge \dots \wedge \mathrm{d}x_\beta^{j_m}\ , \tag{15.41}$$

then according to Example 7 of Sect. 15.1 we find that

$$\sum_{1 \le i_1 < \dots < i_m \le n} a_{i_1 \dots i_m}\, \mathrm{d}x_\alpha^{i_1} \wedge \dots \wedge \mathrm{d}x_\alpha^{i_m} =$$

$$= \sum_{\substack{1 \le i_1 < \dots < i_m \le n \\ 1 \le j_1 < \dots < j_m \le n}} b_{j_1 \dots j_m} \frac{\partial\big(x_\beta^{j_1}, \dots, x_\beta^{j_m}\big)}{\partial\big(x_\alpha^{i_1}, \dots, x_\alpha^{i_m}\big)}(x_\alpha)\, \mathrm{d}x_\alpha^{i_1} \wedge \dots \wedge \mathrm{d}x_\alpha^{i_m}\ , \tag{15.42}$$

where $\frac{\partial(\ )}{\partial(\ )}$, as always, denotes the determinant of the matrix of corresponding partial derivatives.

Thus different coordinate expressions for the same form ω can be obtained from each other by direct substitution of the variables (expanding the corresponding differentials of the coordinates followed by algebraic transformations in accordance with the laws of exterior products).

If we agree to regard the form ω_α as the transfer of a form ω defined on a manifold to the parameter domain of the chart $(U_\alpha, \varphi_\alpha)$, it is natural to write $\omega_\alpha = \varphi_\alpha^* \omega$ and consider that $\omega_\alpha = \varphi_\alpha^* \circ (\varphi_\beta^{-1})^* \omega_\beta = \varphi_{\alpha\beta}^* \omega_\beta$, where the composition $\varphi_\alpha^* \circ (\varphi_\beta^{-1})^*$ in this case plays the role of a formal elaboration of the mapping $\varphi_{\alpha\beta}^* = (\varphi_\beta^{-1} \circ \varphi_\alpha)^*$.

Definition 5. A differential m-form ω on an n-dimensional manifold M is a $C^{(k)}$ *form* if the coefficients $a_{i_1 \dots i_m}(x_\alpha)$ of its coordinate representation

$$\omega_\alpha = \varphi_\alpha^* \omega = \sum_{1 \le i_1 < \cdots < i_m \le n} a_{i_1 \dots i_m}(x_\alpha)\, \mathrm{d}x_\alpha^{i_1} \wedge \cdots \wedge \mathrm{d}x_\alpha^{i_m}$$

are $C^{(k)}$ functions in every chart $(U_\alpha, \varphi_\alpha)$ of an atlas that defines a smooth structure on M.

It is clear from (15.42) that Definition 5 is unambiguous if the manifold M itself is a $C^{(k+1)}$ manifold, for example if M is a $C^{(\infty)}$ manifold.

For differential forms defined on a manifold the operations of addition, multiplication by a scalar, and exterior multiplication are naturally defined pointwise. (In particular, multiplication by a function $f : M \to \mathbb{R}$, which by definition is regarded as a form of degree zero, is defined.) The first two of these operations turn the set Ω_k^m of m-forms of class $C^{(k)}$ on M into a vector space. In the case $k = \infty$ this vector space is usually denoted Ω^m. It is clear that exterior multiplication of forms $\omega^{m_1} \in \Omega_k^{m_1}$ and $\omega^{m_2} \in \Omega_k^{m_2}$ yields a form $\omega^{m_1+m_2} = \omega^{m_1} \wedge \omega^{m_2} \in \Omega_k^{m_1+m_2}$.

15.3.3 The Exterior Derivative

Definition 6. The *exterior differential* is the linear operator $\mathrm{d} : \Omega_k^m \to \Omega_{k-1}^{m+1}$ possessing the following properties:

1^0 On every function $f \in \Omega_k^0$ the differential $\mathrm{d} : \Omega_k^0 \to \Omega_{k-1}^1$ equals the usual differential $\mathrm{d}f$ of this function.

2^0 $\mathrm{d} : (\omega^{m_1} \wedge \omega^{m_2}) = \mathrm{d}\omega^{m_1} \wedge \omega^{m_2} + (-1)^{m_1} \omega^{m_1} \wedge \mathrm{d}\omega^{m_2}$, where $\omega^{m_1} \in \Omega_k^{m_1}$ and $\omega^{m_2} \in \Omega_k^{m_2}$.

3^0 $\mathrm{d}^2 := \mathrm{d} \circ \mathrm{d} = 0$.

This last equality means that $\mathrm{d}(\mathrm{d}\omega)$ is zero for every form ω.

Requirement 3^0 thus presumes that we are talking about forms whose smoothness is at least $C^{(2)}$.

In practice this means that we are considering a $C^{(\infty)}$ manifold M and the operator d mapping Ω^m to Ω^{m+1}.

A formula for computing the operator d in local coordinates of a specific chart (and at the same time the uniqueness of the operator d) follows from the relation

$$\mathrm{d}\Big(\sum_{1\le i_1<\cdots<i_m\le n} c_{i_1\dots i_m}(x)\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_m}\Big) = \\ = \sum_{1\le i_1<\cdots<i_m\le n} \mathrm{d}c_{i_1\dots i_m}(x)\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_m} + \\ + \Big(\sum_{1\le i_1<\cdots<i_m\le n} c_{i_1,\dots,i_m}\,\mathrm{d}(\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_m})=0\Big)\,. \tag{15.43}$$

The existence of the operator d now follows from the fact that the operator defined by (15.43) in a local coordinate system satisfies conditions 1^0, 2^0, and 3^0 of Definition 6.

It follows in particular from what has been said that if $\omega_\alpha = \varphi_\alpha^*\omega$ and $\omega_\beta = \varphi_\beta^*\omega$ are the coordinate representations of the same form ω, that is, $\omega_\alpha = \varphi_{\alpha\beta}^*\omega_\beta$, then $\mathrm{d}\omega_\alpha$ and $\mathrm{d}\omega_\beta$ will also be the coordinate representations of the same form $(\mathrm{d}\omega)$, that is, $\mathrm{d}\omega_\alpha = \varphi_{\alpha\beta}^*\mathrm{d}\omega_\beta$. Thus the relation $\mathrm{d}(\varphi_{\alpha\beta}^*\omega_\beta) = \varphi_{\alpha\beta}^*(\mathrm{d}\omega_\beta)$ holds, which in abstract form asserts the commutativity

$$\mathrm{d}\varphi^* = \varphi^*\mathrm{d} \tag{15.44}$$

of the operator d and the operation φ^* that transfers forms.

15.3.4 The Integral of a Form over a Manifold

Definition 7. Let M be an n-dimensional smooth oriented manifold on which the coordinates $x^1,\dots,x^n$ and the orientation are defined by a single chart $\varphi_x : D_x \to M$ with parameter domain $D_x \subset \mathbb{R}^n$. Then

$$\int_M \omega := \int_{D_x} a(x)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\,, \tag{15.45}$$

where the left-hand side is the usual *integral of the form ω over the oriented manifold M* and the right-hand side is the integral of the function $f(x)$ over the domain D_x.

If $\varphi_t : D_t \to M$ is another atlas of M consisting of a single chart defining the same orientation on M as $\varphi_x : D_x \to M$, then the Jacobian $\det\varphi'(t)$ of the function $x = \varphi(t)$ of the coordinate change is everywhere positive in D_t. The form

$$\varphi^*\big(a(x)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\big) = a\big(x(t)\big)\,\det\varphi'(t)\,\mathrm{d}t^1\wedge\cdots\wedge\mathrm{d}t^n$$

in D_t corresponds to the form ω. By the theorem on change of variables in a multiple integral we have the equality

$$\int_{D_x} a(x)\,\mathrm{d}x^1 \cdots \mathrm{d}x^n = \int_{D_t} a\bigl(x(t)\bigr) \det \varphi'(t)\,\mathrm{d}t^1 \cdots \mathrm{d}t^n ,$$

which shows that the left-hand side of (15.45) is independent of the coordinate system chosen in M.

Thus, Definition 7 is unambiguous.

Definition 8. The *support* of a form ω defined on a manifold M is the closure of the set of points $x \in M$ where $\omega(x) \neq 0$.

The support of a form ω is denoted by $\operatorname{supp}\omega$. In the case of 0-forms, that is, functions, we have already encountered this concept. Outside the support the coordinate representation of the form in any local coordinate system is the zero form of the corresponding degree.

Definition 9. A form ω defined on a manifold M is *of compact support* if $\operatorname{supp}\omega$ is a compact subset of M.

Definition 10. Let ω be a form of degree n and compact support on an n-dimensional smooth manifold M oriented by the atlas A. Let $\varphi_i : D_i \to U_i$, $\{(U_i, \varphi), i = 1, \dots, m\}$ be a finite set of charts of the atlas A whose ranges $U_1, \dots, U_m$ cover $\operatorname{supp}\omega$, and let $e_1, \dots, e_k$ be a partition of unity subordinate to that covering on $\operatorname{supp}\omega$. Repeating some charts several times if necessary, we can assume that $m = k$, and that $\operatorname{supp} e_i \subset U_i$, $i = 1, \dots, m$.

The *integral of a form ω of compact support over the oriented manifold M* is the quantity

$$\int_M \omega := \sum_{i=1}^{m} \int_{D_i} \varphi_i^*(e_i\omega) , \tag{15.46}$$

where $\varphi_i^*(e_i\omega)$ is the coordinate representation of the form $e_i\omega\big|_{U_i}$ in the domain D_i of variation of the coordinates of the corresponding local chart.

Let us prove that this definition is unambiguous.

Proof. Let $\widetilde{A} = \{\widetilde{\varphi}_j : \widetilde{D}_j \to \widetilde{U}_j\}$ be a second atlas defining the same smooth structure and orientation on M as the atlas A, let $\widetilde{U}_1, \dots, \widetilde{U}_{\widetilde{m}}$ be the corresponding covering of $\operatorname{supp}\omega$, and let $\widetilde{e}_1, \dots, \widetilde{e}_{\widetilde{m}}$ a partition of unity on $\operatorname{supp}\omega$ subordinate to this covering. We introduce the functions $f_{ij} = e_i\widetilde{e}_j$, $i = 1, \dots, m$, $j = 1, \dots, \widetilde{m}$, and we set $\omega_{ij} = f_{ij}\omega$.

We remark that $\operatorname{supp}\omega_{ij} \subset W_{ij} = U_i \cap \widetilde{U}_j$. From this and from the fact that Definition 7 of the integral over an oriented manifold given by a single chart is unambiguous it follows that

$$\int_{D_i} \varphi_i^*(\omega_{ij}) = \int_{\varphi_i^{-1}(W_{ij})} \varphi_i^*(\omega_{ij}) = \int_{\widetilde{\varphi}_j^{-1}(W_{ij})} \widetilde{\varphi}_j^*(\omega_{ij}) = \int_{\widetilde{D}_j} \widetilde{\varphi}_j^*(\omega_{ij}) .$$

Summing these equalities on i from 1 to m and on j from 1 to $\widetilde{m}$, taking account of the relation $\sum\limits_{i=1}^{m} f_{ij} = \widetilde{e}_j$, $\sum\limits_{j=1}^{\widetilde{m}} f_{ij} = e_i$, we find the identities we are interested int. □

15.3.5 Stokes' Formula

Theorem. *Let M be an oriented smooth n-dimensional manifold and ω a smooth differential form of degree $n-1$ and compact support on M. Then*

$$\int\limits_{\partial M} \omega = \int\limits_{M} \mathrm{d}\omega\,, \tag{15.47}$$

where the orientation of the boundary ∂M of the manifold M is induced by the orientation of the manifold M. If $\partial M = \varnothing$, then $\int\limits_{M} \mathrm{d}\omega = 0$.

Proof. Without loss of generality we may assume that the domains of variation of the coordinates (parameters) of all local charts of the manifold M are either the open cube $I = \{x \in \mathbb{R}^n \big| 0 < x^i < 1,\, i = 1, \dots, n\}$, or the cube $\tilde{I} = \{x \in \mathbb{R}^n \big| 0 < x^1 \le 1 \wedge 0 < x^i < 1,\, i = 1, \dots, n\}$ with one (definite!) face adjoined to the cube I.

By the partition of unity the assertion of the theorem reduces to the case when $\operatorname{supp}\omega$ is contained in the range U of a single chart of the form $\varphi : I \to U$ or $\varphi : \tilde{I} \to U$. In the coordinates of this chart the form ω has the form

$$\omega = \sum_{i=1}^{n} a_i(x)\,\mathrm{d}x^1 \wedge \dots \wedge \widehat{\mathrm{d}x^i} \wedge \dots \wedge \mathrm{d}x^n\,,$$

where the frown $\frown$, as usual, means that the corresponding factor is omitted.

By the linearity of the integral, it suffices to prove the assertion for one term of the sum:

$$\omega_i = a_i(x)\,\mathrm{d}x^1 \wedge \dots \wedge \widehat{\mathrm{d}x^i} \wedge \dots \wedge \mathrm{d}x^n\,. \tag{15.48}$$

The differential of such a form is the n-form

$$\mathrm{d}\omega_i = (-1)^{i-1} \frac{\partial a_i}{\partial x^i}(x)\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n\,. \tag{15.49}$$

For a chart of the form $\varphi : I \to U$ both integrals in (15.47) of the corresponding forms (15.48) and (15.49) are zero: the first because $\operatorname{supp} a_i \subset I$ and the second for the same reason, if we take into account Fubini's theorem and the relation $\int\limits_0^1 \frac{\partial a_i}{\partial x^i}\,\mathrm{d}x^i = a_i(1) - a_i(0) = 0$. This argument also covers the case when $\partial M = \varnothing$.

Thus it remains to verify (15.47) for a chart $\varphi : \tilde{I} \to U$.

If $i > 1$, both integrals are also zero for such a chart, which follows from the reasoning given above.

And if $i = 1$, then

$$\int_M \omega_1 = \int_U \omega_1 = \int_{\tilde{I}} \frac{\partial a_1}{\partial x^1}(x)\,\mathrm{d}x^1 \cdots \mathrm{d}x^n =$$

$$= \int_0^1 \cdots \int_0^1 \left(\int_0^1 \frac{\partial a_1}{\partial x^1}(x)\,\mathrm{d}x^1 \right) \mathrm{d}x^2 \cdots \mathrm{d}x^n =$$

$$= \int_0^1 \cdots \int_0^1 a_1(1, x^2, \cdots, x^n)\,\mathrm{d}x^2 \cdots \mathrm{d}x^n = \int_{\partial U} \omega_1 = \int_{\partial M} \omega_1 .$$

Thus formula (15.47) is proved for $n > 1$.

The case $n = 1$ is merely the Newton–Leibniz formula (the fundamental theorem of calculus), if we assume that the endpoints α and β of the oriented interval $[\alpha, \beta]$ are denoted α_- and β_+ and the integral of a 0-form $g(x)$ over such an oriented point is equal to $-g(\alpha)$ and $+g(\beta)$ respectively. □

We now make some remarks on this theorem.

Remark 1. Nothing is said in the statement of the theorem about the smoothness of the manifold M and the form ω. In such cases one usually assumes that each of them is $C^{(\infty)}$. It is clear from the proof of the theorem, however, that formula (15.47) is also true for forms of class $C^{(2)}$ on a manifold M admitting a form of this smoothness.

Remark 2. It is also clear from the proof of the theorem, as in fact it was already from the formula (15.47), that if $\operatorname{supp}\omega$ is a compact set contained strictly inside M, that is, $\operatorname{supp}\omega \cap \partial M = \varnothing$, then $\int\limits_M \mathrm{d}\omega = 0$.

Remark 3. If M is a compact manifold, then for every form ω on M the support $\operatorname{supp}\omega$, being a closed subset of the compact set M, is compact. Consequently in this case every form ω on M is of compact support and Eq. (15.47) holds. In particular, if M is a compact manifold without boundary, then the equality $\int\limits_M \mathrm{d}\omega = 0$ holds for every smooth form on M.

Remark 4. For arbitrary forms ω (not of compact support) on a manifold that is not itself compact, formula (15.47) is in general not true.

Let us consider, for example, the form $\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$ in a circular annulus $M = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \le x^2 + y^2 \le 2\}$, endowed with standard Cartesian coordinates. In this case M is a compact two-dimensional oriented manifold,